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On absolute continuity of the spectrum of a periodic magnetic Schrödinger operator

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Abstract

We consider the Schrödinger operator in $\mathbb{R}^n$, $n \geqslant 3$, with the electric potential V and the magnetic potential A being periodic functions (with a common period lattice) and prove absolute continuity of the spectrum of the operator in question under some conditions which, in particular, are satisfied if $V \in L^{n/2}_{\mathrm{loc}}(\mathbb{R}^n)$ and $A \in H^q_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{R}^n)$, $q > (n-1)/2$.

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The paper concerns the problem of absolute continuity of the spectrum of a periodic magnetic Schrödinger operator. Periodic elliptic differential operators arise in many areas of mathematical physics. The stationary Schrödinger operator

$$-\Delta + V(x), \qquad x \in \mathbb{R}^n, \tag{1}$$

with a periodic electric potential V plays an important role in the quantum solid state theory (see, e.g., [1, 2]). We should also mention the periodic Maxwell operator (see [3–5]), the generalized periodic magnetic Schrödinger operator

$$\sum_{j,l=1}^{n} \left(-\mathrm{i}\frac{\partial}{\partial x_j} - A_j\right) G_{jl} \left(-\mathrm{i}\frac{\partial}{\partial x_l} - A_l\right) + V, \qquad x \in \mathbb{R}^n, \tag{2}$$

with the electric potential V and the magnetic potential A, where $\{G_{jl}\}$ is a positive definite matrix function (see [6]), and the periodic Dirac operator (see, e.g., [7–9] and also [10, 11]). The operator (2) for $A \equiv 0$ and $V \equiv 0$ is also used in studying periodic acoustic media.

It is well known that the spectra of periodic elliptic operators have a band-gap structure. In [12], for the periodic electric potential $V \in L^2_{\mathrm{loc}}(\mathbb{R}^3)$, Thomas proved absolute continuity of the spectrum of operator (1) on $L^2(\mathbb{R}^3)$. In particular, this means that the spectrum of operator (1) does not contain any eigenvalues, hence the spectral bands do not collapse to a point. In [13, 14], it was proved that the singular continuous part is missing from the spectra of periodic elliptic operators. Therefore absolute continuity of the spectra of these operators is equivalent to the absence of eigenvalues.

In [15], Filonov presented examples of periodic operators

$$\sum_{j,l=1}^{n}\left(-\mathrm{i}\frac{\partial}{\partial x_j}\right)G_{jl}\left(-\mathrm{i}\frac{\partial}{\partial x_l}\right)$$

in $\mathbb{R}^n$, $n \geqslant 3$, whose spectra have eigenvalues (of infinite multiplicity), where $\{G_{jl}\}$ are some positive definite periodic matrix functions which belong to all Hölder classes C^{α}, $\alpha < 1$.

Since eigenfunctions corresponding to eigenvalues are considered as bound states and ones that correspond to the absolutely continuous spectrum are interpreted as propagating modes, the absolute continuity of the spectrum is a physically important property. In the last decade many papers were devoted to the problem of absolute continuity of the spectra of periodic elliptic operators. The papers [6, 16–18] contain a survey of relevant results.

In this paper we consider the periodic Schrödinger operator

$$\widehat{H}(A,V)=\sum_{j=1}^{n}\left(-\mathrm{i}\frac{\partial}{\partial x_j}-A_j\right)^2+V \tag{3}$$

acting on $L^2(\mathbb{R}^n)$, $n \geqslant 2$, where the electric potential $V:\mathbb{R}^n\to\mathbb{R}$ and the magnetic potential $A:\mathbb{R}^n\to\mathbb{R}^n$ are periodic functions with a common period lattice $\Lambda\subset\mathbb{R}^n$.

The coordinates in $\mathbb{R}^n$ are taken relative to an orthogonal basis $\{\mathcal{E}_j\}$. Let K be the fundamental domain of the lattice Λ, $\{E_j\}$ the basis in the lattice Λ, Λ^* the reciprocal lattice with the basis vectors E_j^* satisfying the conditions $(E_j^*,E_l)=\delta_{jl}$ (where δ_{jl} is the Kronecker delta).

The scalar products and the norms on the spaces $\mathbb{C}^M$, $L^2(\mathbb{R}^n;\mathbb{C}^M)$ and $L^2(K;\mathbb{C}^M)$, where $M\in\mathbb{N}$, are introduced in the usual way (as a rule, omitting the notation for the corresponding space). We suppose that the scalar products are linear in the second argument. Let $H^q(\mathbb{R}^n;\mathbb{C}^M)$, $q\geqslant 0$, be the Sobolev class, $\widetilde{H}^q(K;\mathbb{C}^M)$ the set of functions $\phi:K\to\mathbb{C}^M$ whose Λ-periodic extensions belong to $H^q_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{C}^M)$; $\widetilde{H}^q(K)\doteq\widetilde{H}^q(K;\mathbb{C})$. In what follows, the functions defined on the fundamental domain K will be also identified with their Λ-periodic extensions to $\mathbb{R}^n$. We let

$$\phi_N=v^{-1}(K)\int_K\phi(x)\,\mathrm{e}^{-2\pi\mathrm{i}(N,x)}\,\mathrm{d}x,\qquad N\in\Lambda^*,$$

denote the Fourier coefficients of the functions $\phi\in L^1(K;\mathbb{C}^M)$, $v(\cdot)$ is the Lebesgue measure on $\mathbb{R}^n$.

Let $\|\cdot\|_p$ be the norm on the space $L^p(K)$, $p\geqslant 1$. Denote by $L^p_w(K)$ the space of measurable functions $\mathcal{W}:K\to\mathbb{C}$ which satisfy the condition

$$\|\mathcal{W}\|_{p,w}\doteq\sup_{t>0}t(v(\{x\in K:|\mathcal{W}(x)|>t\}))^{1/p}<+\infty.$$

For $\mathcal{W}\in L^p_w(K)$, we also write

$$\|\mathcal{W}\|^{(\infty)}_{p,w}\doteq\overline{\sup_{t\to+\infty}}\,t(v(\{x\in K:|\mathcal{W}(x)|>t\}))^{1/p};$$

$L^p_{w,0}(K)=\left\{\mathcal{W}\in L^p_w(K):\|\mathcal{W}\|^{(\infty)}_{p,w}=0\right\}$.

In the following, we assume that the form $(\phi,V\phi)$, $\phi\in H^1(\mathbb{R}^n)$, has a bound less than 1 relative to the form $\sum_j\left\|\frac{\partial\phi}{\partial x_j}\right\|^2$, $\phi\in H^1(\mathbb{R}^n)$ (in particular, it is true if $V\in L^{n/2}_w(K)$ and $\|V\|^{(\infty)}_{n/2,w}$ is sufficiently small) and for the magnetic potential A the estimate

$$\||A|\phi\|\leqslant\varepsilon\left(\sum_{j=1}^{n}\left\|\frac{\partial\phi}{\partial x_j}\right\|^2\right)^{1/2}+C_\varepsilon\|\phi\|,\qquad\phi\in H^1(\mathbb{R}^n), \tag{4}$$

holds for any $\varepsilon > 0$, where $C_\varepsilon = C_\varepsilon(n; A) \geqslant 0$. Under these conditions the quadratic form

$$W(A, V; \phi, \phi) = \sum_{j=1}^{n} \left\| \left(-\mathrm{i}\frac{\partial}{\partial x_j} - A_j \right) \phi \right\|^2 + (\phi, V\phi), \qquad \phi \in H^1(\mathbb{R}^n),$$

with the domain $Q(W) = H^1(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$ is closed and semi-bounded from below. Therefore the form W generates the self-adjoint operator (3) with some domain $D(\widehat{H}(A, V)) \subset H^1(\mathbb{R}^n)$.

The problem of absolute continuity of the spectra of two-dimensional periodic Schrödinger operators (2) and (3) has been thoroughly studied (see [19–30]) and optimal conditions on the electric potential V and the magnetic potential A have already been obtained. In particular, for the two-dimensional operator (3), absolute continuity of the spectrum was proved if the form $(\phi, V\phi)$ has a zero bound relative to the form $\sum_j \| \frac{\partial \phi}{\partial x_j} \|^2, \phi \in H^1(\mathbb{R}^2)$, and for the magnetic potential A, estimate (4) holds for all $\varepsilon > 0$ (see [28] and also [30]). In [2], the results of the paper [12] were generalized on n-dimensional Schrödinger operators (1) with the periodic potentials V for which $V \in L^2_{\mathrm{loc}}(\mathbb{R}^n), n = 2, 3$, and $\sum_{N\in\Lambda^*} |V_N|^q < +\infty, 1 \leqslant q < (n-1)/(n-2), n \geqslant 4$. For $n \geqslant 3$, absolute continuity of the spectrum of the Schrödinger operator (3) was established by Sobolev (see [31]) for the periodic potentials $V \in L^p(K), p > n-1$, and $A \in C^{2n+3}(\mathbb{R}^n; \mathbb{R}^n)$. These conditions on the potentials V and A (for $n \geqslant 3$) were relaxed in subsequent papers. In [6], it was supposed that $V \in L^{n/2}_{w,0}(K)$ for $n = 3, 4$ and $V \in L^{n-2}_{w,0}(K)$ for $n \geqslant 5$. In [17, 32], the constraint on the magnetic potential A was relaxed up to $A \in H^q_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{R}^n), 2q > 3n - 2$. In [33, 34], for the magnetic potential $A \in C^1(\mathbb{R}^n; \mathbb{R}^n)$ it was assumed that either $A \in H^q_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{R}^n), 2q > n-2$, or $\sum_{N\in\Lambda^*} \|A_N\|_{\mathbb{C}^n} < +\infty$, and $V \in L^p_w(K), \|V\|^{(\infty)}_{p,w} \leqslant C'$, where $p = n/2$ for $n = 3, 4, 5, 6$ and $p = n - 3$ for $n \geqslant 7, C' = C'(n, \Lambda; A) > 0$. The absolute continuity of the spectrum of the Schrödinger operator with the periodic potential $V \in L^{n/2}_w(K)$ for which $\|V\|^{(\infty)}_{n/2,w}$ is sufficiently small was proved in [35] for all $n \geqslant 3$ (and for $A \equiv 0$). The periodic electric potentials V from the Kato class and from the Morrey class were also considered in [24, 36], respectively. For $n \geqslant 3$, the periodic Schrödinger operator (3) and its generalization (2) were also considered in [37–41]. In [24, 35, 36], for $n \geqslant 3$ and $A \equiv 0$, the optimal conditions on the periodic electric potential V were approached in terms of standard functional spaces (but it is believed that the known conditions on the periodic magnetic potential A are not optimal for $n \geqslant 3$). In theorem 0.1 we relax conditions on the periodic potentials V and A. If the periodic Schrödinger operator (3) has the period lattice $\Lambda = \mathbb{Z}^n, n \geqslant 3$, and is invariant under the substitution $x_1 \to -x_1$, then its spectrum is absolutely continuous under the conditions $A \in L^q_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{R}^n), q > n$ and $V \in L^{n/2}_{\mathrm{loc}}(\mathbb{R}^n)$ (see [42]).

For the vectors $x \in \mathbb{R}^n \backslash \{0\}$ we shall use the notation

$$S_{n-2}(x) = \{\widetilde{e} \in S_{n-1} : (\widetilde{e}, x) = 0\},$$

where $S_{n-1} = \{y \in \mathbb{R}^n : |y| = 1\}$.

Let $\mathcal{B}(\mathbb{R})$ be the collection of Borel subsets $\mathcal{O} \subseteq \mathbb{R}$, $\mathfrak{M}$ the set of even signed Borel measures $\mu : \mathcal{B}(\mathbb{R}) \to \mathbb{R}$,

$$\|\mu\| = \sup_{\mathcal{O}\in\mathcal{B}(\mathbb{R})} (|\mu(\mathcal{O})| + |\mu(\mathbb{R}\backslash\mathcal{O})|) < +\infty, \qquad \mu \in \mathfrak{M}.$$

Denote by $\mathfrak{M}_h, h > 0$, the set of measures $\mu \in \mathfrak{M}$ such that

$$\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}pt} \, \mathrm{d}\mu(t) = 1$$

for all $p \in (-h, h)$. In particular, the set $\mathfrak{M}_h$ contains the Dirac measure $\delta(\cdot)$.

The following theorem is the main result of this paper.

Theorem 0.1. *Let $n \geqslant 3$ and let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a periodic function with a period lattice $\Lambda \subset \mathbb{R}^n$. Fix a vector $\gamma \in \Lambda\backslash\{0\}$. Suppose that the magnetic potential $A \in L^2(K;\mathbb{R}^n)$ satisfies the following two conditions:*

(A_1) the map

$$\mathbb{R}^n \ni x \to \{[0,1] \ni \xi \to A(x-\xi\gamma)\} \in L^2([0,1];\mathbb{R}^n)$$

is continuous;

(A_2) there is a measure $\mu \in \mathfrak{M}_h$, $h > 0$, such that

$$\theta(\Lambda,\gamma,h,\mu;A) \doteq \frac{|\gamma|}{\pi} \max_{x\in\mathbb{R}^n} \max_{\widetilde{e}\in S_{n-2}(\gamma)} \left| A_0 - \int_{\mathbb{R}} \mathrm{d}\mu(t) \int_0^1 A(x-\xi\gamma-t\widetilde{e})\,\mathrm{d}\xi \right| < 1, \quad (5)$$

where $A_0 = v^{-1}(K)\int_K A(x)\,\mathrm{d}x$ (and $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^n$).

Then there exists a number $C = C(n,\Lambda;A) > 0$ such that for all electric potentials $V = V_1 + V_2$, where $V_1 \in L^{n/2}_w(K;\mathbb{R})$ and $V_2 \in L^1(K;\mathbb{R})$ are Λ-periodic functions for which

$$\|V_1\|^{(\infty)}_{n/2,w} \leqslant C \quad (6)$$

and

$$\operatorname*{ess\,sup}_{x\in\mathbb{R}^n} \int_0^1 |V_2(x-\xi\gamma)|\,\mathrm{d}\xi < +\infty, \quad (7)$$

the spectrum of the periodic Schrödinger operator (3) is absolutely continuous.

Theorem 0.1 is proved in section 1.

Remark 1. Under the conditions of theorem 0.1, the number $C = C(n,\Lambda;A)$ in inequality (6) is chosen sufficiently small so that the form $(\phi, V_1\phi)$ has a bound less than 1 relative to the form $\sum_j \left\|\frac{\partial\phi}{\partial x_j}\right\|^2$, $\phi \in H^1(\mathbb{R}^n)$. Furthermore, from (7) it follows that the form $(\phi, V_2\phi)$ has a zero bound relative to the form $\sum_j \left\|\frac{\partial\phi}{\partial x_j}\right\|^2$, and the condition (A_1) implies that inequality (4) holds for all $\varepsilon > 0$. Hence the periodic Schrödinger operator (3) is generated by the quadratic form $W(A,V;\phi,\phi)$, $\phi \in H^1(\mathbb{R}^n)$, which is closed and semi-bounded from below.

Remark 2. Instead of condition (6) one can admit the weakened condition

$$\lim_{r\to+0} \sup_{x\in\mathbb{R}^n} \overline{\sup_{t\to+\infty}}\, t(v(\{y \in B_r(x) : |V_1(y)| > t\}))^{2/n} \leqslant C$$

(with another constant $C = C(n,\Lambda;A) > 0$), where $B_r(x) = \{y \in \mathbb{R}^n : |x-y| \leqslant r\}$ is a closed ball of radius $r > 0$ centered at $x \in \mathbb{R}^n$.

Remark 3. For the periodic magnetic potential A the condition (A_2) is fulfilled (under an appropriate choice of the vector $\gamma \in \Lambda\backslash\{0\}$ and the measure $\mu \in \mathfrak{M}_h$, $h > 0$) if $A \in \widetilde{H}^q(K;\mathbb{R}^n)$, $2q > n-2$ (see [11, 33]). If $2q > n-1$, then the condition (A_1) is fulfilled as well. For the choice of the Dirac measure $\mu = \delta$ in the condition (A_2), inequality (5) is valid whenever

$$\sum_{N\in\Lambda^*\backslash\{0\}:(N,\gamma)=0} \|A_N\|_{\mathbb{C}^n} < \frac{\pi}{|\gamma|}. \quad (8)$$

Moreover, inequality (8) holds under an appropriate choice of the vector $\gamma \in \Lambda\backslash\{0\}$ if $\sum_{N\in\Lambda^*} \|A_N\|_{\mathbb{C}^n} < +\infty$ (see [11, 33]).

The proof of theorem 0.1 follows the method suggested by Thomas in [12]. In this paper we apply estimates for the periodic electric potential $V_1 \in L^{n/2}_w(K;\mathbb{R})$ (see (13) and theorem 1.2) which are derived as a consequence of the Tomas–Stein inequality for the restriction of the Fourier transform to the unit sphere (see a survey on such estimates in [43, 44]). Besides, the estimates are obtained for L^2-norms (unlike [35]) so this allows us to study the Schrödinger operator (3) with the magnetic potential A. For the proof of theorem 0.1, we also apply assertions for the periodic magnetic Dirac operator (see theorem 3.1) proved in [45, 46].

The proof of theorem 0.1 is presented in section 1. Theorems 1.2 and 1.3 from section 1 are proved in sections 2 and 3, respectively.

In the paper we use the notation C (with subscripts and superscripts or without them) for constants which are not necessarily the same at each occurrence but we shall explicitly indicate on what these constants depend.

1. Proof of theorem 0.1

For $k \in \mathbb{R}^n$, $e \in S_{n-1}$, and $\varkappa \in \mathbb{R}$, let

$$W(A;k+\mathrm{i}\varkappa e;\psi,\phi) = \sum_{j=1}^{n}\left(\left(-\mathrm{i}\frac{\partial}{\partial x_j} - A_j + k_j - \mathrm{i}\varkappa e_j\right)\psi, \left(-\mathrm{i}\frac{\partial}{\partial x_j} - A_j + k_j + \mathrm{i}\varkappa e_j\right)\phi\right)$$

be a sesquilinear form with the domain $Q(W(A;k+\mathrm{i}\varkappa e;.,.)) = \widetilde{H}^1(K) \subset L^2(K)$. Under the conditions imposed on the potentials A and V, the quadratic form $(\phi, V\phi)$ has a bound less than 1 relative to the forms $W(0;k;\phi,\phi)$, $k \in \mathbb{R}^n$, $\phi \in \widetilde{H}^1(K)$. Therefore,

$$W(A,V;k+\mathrm{i}\varkappa e;\psi,\phi) \doteq W(A;k+\mathrm{i}\varkappa e;\psi,\phi) + (\psi, V\phi), \qquad \psi,\phi \in \widetilde{H}^1(K),$$

is a closed sectorial sesquilinear form generating an m-sectorial operator $\widehat{H}(A;k+\mathrm{i}\varkappa e)+V$ (with the domain $D(\widehat{H}(A;k+\mathrm{i}\varkappa e)+V) \subset \widetilde{H}^1(K) \subset L^2(K)$ independent of the complex vector $k+\mathrm{i}\varkappa e \in \mathbb{C}^n$). If $A \in C^1(\mathbb{R}^n;\mathbb{R}^n)$, then

$$\widehat{H}(A;k+\mathrm{i}\varkappa e) = \sum_{j=1}^{n}\left(-\mathrm{i}\frac{\partial}{\partial x_j} - A_j + k_j + \mathrm{i}\varkappa e_j\right)^2$$

and $D(\widehat{H}(A;k+\mathrm{i}\varkappa e)) = \widetilde{H}^2(K)$. The operators $\widehat{H}(A;k)+V$ (for $\varkappa = 0$) are self-adjoint and have compact resolvent. This implies that they have a discrete spectrum. For fixed vectors $k \in \mathbb{R}^n$ and $e \in S_{n-1}$, the operators $\widehat{H}(A;k+\zeta e)+V$, $\zeta \in \mathbb{C}$, form a self-adjoint analytic family of type (B) (see [47]).

The operator $\widehat{H}(A,V)$ is unitarily equivalent to the direct integral

$$\int_{2\pi K^*}^{\oplus} (\widehat{H}(A;k)+V)\frac{\mathrm{d}k}{(2\pi)^n v(K^*)}, \tag{9}$$

where K^* is the fundamental domain of the lattice Λ^*. The unitary equivalence is established via the Gel'fand transformation (see [6, 35]). Let $\lambda_j(k)$, $j \in \mathbb{N}$, be the eigenvalues of the operators $\widehat{H}(A;k)+V$ arranged in non-decreasing order with the multiplicity. The spectrum of the operator $\widehat{H}(A;V)$ has a band-gap structure and consists of the union of the ranges $\{\lambda_j(k) : k \in 2\pi K^*\}$ of the band functions $\lambda_j(k)$, $j \in \mathbb{N}$, which are continuous and piecewise analytic. The singular spectrum of the operator (3) is empty (see [13, 14] and for an elementary proof of this fact also see [48, 49]) and if $\lambda \in \mathbb{R}$ is an eigenvalue of the operator $\widehat{H}(A,V)$, then the decomposition of the operator $\widehat{H}(A,V)$ into the direct integral (9) implies that the number λ is an eigenvalue of the operators $\widehat{H}(A;k)+V$ for a positive measure set of vectors $k \in 2\pi K^*$

(i.e. $v(\{k \in 2\pi K^* : \lambda_j(k) = \lambda\}) > 0$ for some $j \in \mathbb{N}$). Therefore, by analytic Fredholm theorem, it follows that the number λ is an eigenvalue of the operators $\widehat{H}(A; k + \mathrm{i}\varkappa e) + V$ for all $k + \mathrm{i}\varkappa e \in \mathbb{C}^n$ (see [13, 18]). Hence, to prove absolute continuity of the spectrum of operator (3), it suffices for any $\lambda \in \mathbb{R}$ to find vectors $k \in \mathbb{R}^n$, $e \in S_{n-1}$ and a number $\varkappa \geqslant 0$ such that the number λ is not an eigenvalue of the operator $\widehat{H}(A; k + \mathrm{i}\varkappa e) + V$. Since the operators $\widehat{H}(A; k + \mathrm{i}\varkappa e) + V$ are generated by the forms $W(A, V; k + \mathrm{i}\varkappa e; \psi, \phi)$, $\psi, \phi \in \widetilde{H}^1(K)$ (i.e.$(\psi, (\widehat{H}(A; k + \mathrm{i}\varkappa e) + V)\phi) = W(A, V; k + \mathrm{i}\varkappa e; \psi, \phi)$ for all $\psi \in \widetilde{H}^1(K)$ and $\phi \in D(\widehat{H}(A; k + \mathrm{i}\varkappa e) + V) \subset \widetilde{H}^1(K)$), we conclude that theorem 0.1 follows from theorem 1.1 in which for a given vector $\gamma \in \Lambda\backslash\{0\}$ (in particular) it is proved that for any $\lambda \in \mathbb{R}$ the operators $\widehat{H}(A; k + \mathrm{i}\varkappa|\gamma|^{-1}\gamma) + V - \lambda$ are invertible for all vectors $k \in \mathbb{R}^n$ with $|(k, \gamma)| = \pi$, and all sufficiently large numbers $\varkappa > 0$ (dependent on $\lambda \in \mathbb{R}$).

Fix a vector $\gamma \in \Lambda\backslash\{0\}$; $e = |\gamma|^{-1}\gamma \in S_{n-1}$. For vectors $x \in \mathbb{R}^n$ we write $x_\parallel \doteq (x, e)$, $x_\perp \doteq x - (x, e)e$. For all $N \in \Lambda^*$, $k \in \mathbb{R}^n$, and $\varkappa \geqslant 0$, introduce the notation

$$G_N^\pm = G_N^\pm(k + \mathrm{i}\varkappa e) \doteq (|k_\parallel + 2\pi N_\parallel|^2 + (\varkappa \pm |k_\perp + 2\pi N_\perp|)^2)^{1/2}.$$

If $|(k, \gamma)| = \pi$, then $G_N^- \geqslant \pi|\gamma|^{-1}$, $G_N^+ \geqslant \varkappa$, and $G_N^+ G_N^- \geqslant 2\pi|\gamma|^{-1}\varkappa$. The equality

$$\widehat{H}(0; k + \mathrm{i}\varkappa e)\phi = \sum_{N\in\Lambda^*} (k + 2\pi N + \mathrm{i}\varkappa e)^2 \phi_N \, \mathrm{e}^{2\pi \mathrm{i}(N,x)}, \qquad \phi \in \widetilde{H}^2(K),$$

holds, where $|(k + 2\pi N + \mathrm{i}\varkappa e)^2| = G_N^+ G_N^-$. Denote by $\widehat{L} = \widehat{L}(k + \mathrm{i}\varkappa e)$ the nonnegative operator on $L^2(K)$

$$\widehat{L}\phi = \sum_{N\in\Lambda^*} G_N^+ G_N^- \phi_N \, \mathrm{e}^{2\pi \mathrm{i}(N,x)}, \qquad \phi \in D(\widehat{L}) = \widetilde{H}^2(K).$$

For the operator $\widehat{L}^{1/2}$, one has $D(\widehat{L}^{1/2}) = \widetilde{H}^1(K)$.

Theorem 1.1. *Let $n \geqslant 3$. Suppose the periodic magnetic potential $A : \mathbb{R}^n \to \mathbb{R}^n$ with the period lattice $\Lambda \subset \mathbb{R}^n$ satisfies the conditions (A_1) and (A_2) of theorem 0.1 and the function $V_2 \in L^1(K; \mathbb{R})$ obeys condition (7) for the fixed vector $\gamma \in \Lambda\backslash\{0\}$. Then there exist numbers $C = C(n, \Lambda; A) > 0$ and $C' = C'(n, \Lambda; A) > 0$ such that for any function $V_1 \in L_w^{n/2}(K; \mathbb{R})$ with $\|V_1\|_{n/2,w}^{(\infty)} \leqslant C$, and any $\lambda \in \mathbb{R}$ there is a number $\varkappa_0 > 0$ such that for all $\varkappa \geqslant \varkappa_0$, all vectors $k \in \mathbb{R}^n$ with $|(k, \gamma)| = \pi$, and all functions $\phi \in \widetilde{H}^1(K)$ the inequality*

$$\sup_{\psi\in\widetilde{H}^1(K):\|\widehat{L}^{1/2}(k+\mathrm{i}\varkappa e)\psi\|\leqslant 1} |W(A, V_1 + V_2 - \lambda; k + \mathrm{i}\varkappa e; \psi, \phi)| \geqslant C'\|\widehat{L}^{1/2}(k + \mathrm{i}\varkappa e)\phi\|$$

holds.

Theorem 1.1 is a consequence of theorems 1.2 and 1.3 and lemma 1.1.

Theorem 1.2. *Let $n \geqslant 3$. Suppose a Λ-periodic function $\mathcal{W} : \mathbb{R}^n \to \mathbb{R}$ belongs to the space $L_w^n(K)$, $\gamma \in \Lambda\backslash\{0\}$ (and $e = |\gamma|^{-1}\gamma$). Then there are numbers $\widetilde{C} = \widetilde{C}(n) > 0$ and $\varkappa_0 > 0$ such that for all $\varkappa \geqslant \varkappa_0$, all vectors $k \in \mathbb{R}^n$ with $|(k, \gamma)| = \pi$, and all functions $\phi \in \widetilde{H}^1(K)$ the inequality*

$$\|\mathcal{W}\phi\| \leqslant \widetilde{C}\|\mathcal{W}\|_{n,w}\|\widehat{L}^{1/2}(k + \mathrm{i}\varkappa e)\phi\|$$

is fulfilled.

For Λ-periodic functions $\mathcal{V} : \mathbb{R}^n \to \mathbb{R}$ from the space $L^p(K)$, $p = 1, 2$, and for the fixed vector $\gamma \in \Lambda\backslash\{0\}$ we write

$$\|\mathcal{V}\|_{p,\gamma} = \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} \left(\int_0^1 |\mathcal{V}(x - \xi\gamma)|^p \, \mathrm{d}\xi \right)^{1/p}.$$

Theorem 1.3. *Let $n \geqslant 3$, $\mathfrak{a} \geqslant 0$, $\Theta \in [0, 1)$. Suppose the periodic magnetic potential $A : \mathbb{R}^n \to \mathbb{R}^n$ with the period lattice $\Lambda \subset \mathbb{R}^n$ satisfies the conditions (A_1) and (A_2) of theorem 0.1 for the fixed vector $\gamma \in \Lambda\backslash\{0\}(e = |\gamma|^{-1}\gamma)$ and, moreover, $\||A|\|_{2,\gamma} \leqslant \mathfrak{a}$ and $\theta(\Lambda, \gamma, h, \mu; A) \leqslant \Theta$. Then there exist numbers $C_1 = C_1(n, \Lambda, |\gamma|, h, \|\mu\|; \mathfrak{a}, \Theta) > 0$ and $\varkappa_0 > 0$ such that for all $\varkappa \geqslant \varkappa_0$, all vectors $k \in \mathbb{R}^n$ with $|(k, \gamma)| = \pi$, and all functions $\phi \in \widetilde{H}^1(K)$ the estimate*

$$\sup_{\psi\in\widetilde{H}^1(K):\|\widehat{L}^{1/2}(k+\mathrm{i}\varkappa e)\psi\|\leqslant 1} |W(A; k + \mathrm{i}\varkappa e; \psi, \phi)| \geqslant C_1\|\widehat{L}^{1/2}(k + \mathrm{i}\varkappa e)\phi\| \tag{10}$$

holds.

Lemma 1.1. *Let $n \geqslant 2$. Suppose a Λ-periodic function $\mathcal{V} : \mathbb{R}^n \to \mathbb{R}$ belongs to the space $L^2(K)$ (and $\|\mathcal{V}\|_{2,\gamma} < +\infty$, where $\gamma \in \Lambda\backslash\{0\}; e = |\gamma|^{-1}\gamma$). Then for any $\varepsilon > 0$ there is a constant $C_\varepsilon = C_\varepsilon(n, |\gamma|) > 0$ such that for all vectors $k \in \mathbb{R}^n$ and all functions $\phi \in \widetilde{H}^1(K)$ the inequality*

$$\|\mathcal{V}\phi\| \leqslant \|\mathcal{V}\|_{2,\gamma}\left(\varepsilon v^{1/2}(K)\left(\sum_{N\in\Lambda^*}|k_\| + 2\pi N_\||^2\|\phi_N\|^2\right)^{1/2} + C_\varepsilon\|\phi\|\right)$$

holds.

Lemma 1.1 immediately follows from simple estimates for functions from the Sobolev class $H^1_{\mathrm{loc}}(\mathbb{R})$ (see, e.g., [50]).

Proof (Proof of theorem 1.1)**.** If a Λ-periodic function $\mathcal{W} : \mathbb{R}^n \to \mathbb{R}$ belongs to the space $L^\infty(\mathbb{R}^n)$, then the inequality

$$\|\mathcal{W}\phi\| \leqslant \|\mathcal{W}\|_\infty\|\phi\| \leqslant \left(\frac{|\gamma|}{2\pi\varkappa}\right)^{1/2}\|\mathcal{W}\|_\infty\|\widehat{L}^{1/2}(k + \mathrm{i}\varkappa e)\phi\| \tag{11}$$

is fulfilled for all $\varkappa > 0$, all vectors $k \in \mathbb{R}^n$ with $|(k, \gamma)| = \pi$, and all functions $\phi \in \widetilde{H}^1(K)$. By theorem 1.2 and estimate (11), it follows that for a function $\mathcal{W} \in L^n_w(K)$ and for any $\varepsilon > 0$ (assuming the number $\varkappa_0 > 0$ to be sufficiently large) the inequality

$$\|\mathcal{W}\phi\| \leqslant \widetilde{C}\big(\varepsilon^2 + \big(\|\mathcal{W}\|^{(\infty)}_{n,w}\big)^2\big)^{1/2}\|\widehat{L}^{1/2}(k + \mathrm{i}\varkappa e)\phi\| \tag{12}$$

holds for all $\varkappa \geqslant \varkappa_0$, all vectors $k \in \mathbb{R}^n$ with $|(k, \gamma)| = \pi$, and all functions $\phi \in \widetilde{H}^1(K)$. Denoting $\mathcal{W} = \sqrt{|V_1|}$ we have $\mathcal{W} \in L^n_w(K)$ and $\|\mathcal{W}\|^{(\infty)}_{n,w} = \big(\|V_1\|^{(\infty)}_{n/2,w}\big)^{1/2}$. Hence from (12) (for all $\varkappa \geqslant \varkappa_0$, all vectors $k \in \mathbb{R}^n$ with $|(k, \gamma)| = \pi$, and all functions $\psi, \phi \in \widetilde{H}^1(K)$) we get

$$|(\psi, V_1\phi)| \leqslant \widetilde{C}^2\big(\varepsilon^2 + \|V_1\|^{(\infty)}_{n/2,w}\big)\|\widehat{L}^{1/2}(k + \mathrm{i}\varkappa e)\psi\| \cdot \|\widehat{L}^{1/2}(k + \mathrm{i}\varkappa e)\phi\|. \tag{13}$$

By lemma 1.1, for any $\varepsilon > 0$ there is a sufficiently large number $\varkappa_0 > 0$ such that the estimate

$$|(\psi, (V_2 - \lambda)\phi)| \leqslant \varepsilon^2\|V_2 - \lambda\|_{1,\gamma}\|\widehat{L}^{1/2}(k + \mathrm{i}\varkappa e)\psi\| \cdot \|\widehat{L}^{1/2}(k + \mathrm{i}\varkappa e)\phi\| \tag{14}$$

is also valid for all $\lambda \in \mathbb{R}$, all $\varkappa \geqslant \varkappa_0$, all vectors $k \in \mathbb{R}^n$ with $|(k, \gamma)| = \pi$, and all functions $\psi, \phi \in \widetilde{H}^1(K)$. Now, theorem 1.1 is a direct consequence of theorem 1.3 and estimates (13) and (14). Furthermore, we can choose any positive number $C < \sqrt{\frac{C_1}{2}}\widetilde{C}^{-1}$ and put $C' = \frac{C_1}{2}$, where $\widetilde{C}$ and C_1 are constants from theorems 1.2 and 1.3. This completes the proof. □

Remark 4. For the vector $\gamma \in \Lambda\backslash\{0\}$ denote by $\widetilde{\gamma} = \widetilde{\gamma}(\gamma)$ the vector of the lattice Λ such that $\widetilde{\gamma} = t\gamma$, $t > 0$, and $\tau\gamma \notin \Lambda$ for all $\tau \in (0, t)$. Let $\mathfrak{M}_{[0,1]}$ be the set of signed Borel measures defined on Borel subsets of the closed interval [0, 1], and let $\mathbb{R}^n \ni x \to \mu(x; .) \in \mathfrak{M}_{[0,1]}$ be a weakly measurable and Λ-periodic measure-valued function such that

$$(1) \quad \int_0^1 f(\xi+\tau)\mu(x+\tau\widetilde{\gamma}; \mathrm{d}\xi) = \int_0^1 f(\xi)\mu(x; \mathrm{d}\xi)$$

for all $x \in \mathbb{R}^n$, $\tau \in \mathbb{R}$ and all periodic functions $f \in C(\mathbb{R})$ with the period $T = 1$,

$$(2) \quad m(\mu) \doteq \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} \int_0^1 |\mu(x; \mathrm{d}\xi)| < +\infty,$$

where $|\mu(x; .)|$ is the variation of the measure $\mu(x; .)$, $x \in \mathbb{R}^n$. Introduce the sesquilinear form

$$\mathcal{M}(\psi, \phi) = \int_K \mathrm{d}x \int_0^1 \overline{\psi}(x-\xi\widetilde{\gamma})\phi(x-\xi\widetilde{\gamma})\mu(x; \mathrm{d}\xi), \qquad \psi, \phi \in \widetilde{H}^1(K). \tag{15}$$

For any $\varepsilon > 0$, there is a number $\varkappa_0 > 0$ such that for all $\lambda \in \mathbb{R}$, all $\varkappa \geqslant \varkappa_0$, all vectors $k \in \mathbb{R}^n$ with $|(k, \gamma)| = \pi$, and all functions $\psi, \phi \in \widetilde{H}^1(K)$ (by analogy with inequality (14)) we get

$$|\mathcal{M}(\psi, \phi) - \lambda(\psi, \phi)| \leqslant \varepsilon^2 C(\mu, \lambda)\|\widehat{L}^{1/2}(k+\mathrm{i}\varkappa e)\psi\| \cdot \|\widehat{L}^{1/2}(k+\mathrm{i}\varkappa e)\phi\|, \tag{16}$$

where $C(\mu, \lambda) = m(\mu) + |\lambda|$. Consequently, under the conditions of theorem 1.1, instead of the form $(\psi, V_2\phi)$ determined by the function V_2 we can deal with the form $\mathcal{M}(\psi, \phi)$, $\psi, \phi \in \widetilde{H}^1(K)$, determined by the periodic measure-valued function $\mathbb{R}^n \ni x \to \mu(x; .)$. Another conditions on the form (15), for which inequalities (16) are fulfilled (for all $\varepsilon > 0$ and in the case where $\varkappa \geqslant \varkappa_0$, $|(k, \gamma)| = \pi$) with some constants $C(\mu, \lambda) > 0$, can be found (for $n \geqslant 3$) in [38].

2. Proof of theorem 1.2

Let $S_{n-2}[\varkappa] = \{x' \in \mathbb{R}^{n-1} : |x'| = \varkappa\}$, $\varkappa > 0$, $n \geqslant 3$, and let $\sigma_{n-2}^{(\varkappa)}$ be the (invariant) surface measure on the sphere $S_{n-2}[\varkappa]$; $S_{n-2} \doteq S_{n-2}[1]$. Define the numbers $p = p(n) = (2n)/(n+2)$ and $q = q(n) = (2n)/(n-2)$; $1/p + 1/q = 1$. For all functions $\mathcal{F}$ from the Schwartz space $\mathcal{S}(\mathbb{R}^{n-1})$, the following Tomas–Stein estimate is valid:

$$\|\widehat{\mathcal{F}}\|_{L^2(S_{n-2}; d\sigma_{n-2}^{(1)})} \leqslant C\|\mathcal{F}\|_{L^p(\mathbb{R}^{n-1})} \tag{17}$$

(see [51, 52], and for $n = 3$ also see [53]), where $C = C(n) > 0$ and

$$\widehat{\mathcal{F}}(k') = \frac{1}{(2\pi)^{n-1}} \int_{\mathbb{R}^{n-1}} \mathcal{F}(x')\,\mathrm{e}^{-\mathrm{i}(k',x')}\,\mathrm{d}x', \qquad k' \in \mathbb{R}^{n-1},$$

denotes the Fourier transform of the function $\mathcal{F}$. Estimate (17) is a key point in the proof of theorem 1.2.

Let

$$\mathcal{L}_a^{(n-1)} = \{k' \in \mathbb{R}^{n-1} : \varkappa - a \leqslant |k'| \leqslant \varkappa + a\}, \qquad \varkappa > 0, \qquad 0 < a \leqslant \frac{3}{4}\varkappa.$$

For functions $u \in L^2(\mathcal{L}_a^{(n-1)})$, we shall use the notation

$$\breve{u}(x') = \int_{\mathcal{L}_a^{(n-1)}} u(k')\,\mathrm{e}^{\mathrm{i}(k',x')}\,\mathrm{d}k', \qquad x' \in \mathbb{R}^{n-1}.$$

We have $\breve{u} \in C^\infty(\mathbb{R}^{n-1}) \cap L^s(\mathbb{R}^{n-1})$ for all $s \in [2, +\infty]$.

Lemma 2.1. *For any function $u \in L^2(\mathcal{L}_a^{(n-1)})$, the estimate*

$$\|\breve{u}\|_{L^q(\mathbb{R}^{n-1})} \leqslant C_1 a^{1/2}\varkappa^{1/q}\|u\|_{L^2(\mathcal{L}_a^{(n-1)})}$$

holds, where $C_1 = C_1(n) > 0$.

Proof. By (17), for all $\varkappa > 0$ and all $\mathcal{F} \in \mathcal{S}(\mathbb{R}^{n-1})$, we get

$$\left(\int_{S_{n-2}[\varkappa]} |\widehat{\mathcal{F}}|^2 \, \mathrm{d}\sigma_{n-2}^{(\varkappa)} \right)^{1/2} \leqslant C \varkappa^{1/q} \|\mathcal{F}\|_{L^p(\mathbb{R}^{n-1})}. \tag{18}$$

Using (18) one immediately derives

$$\left(\int_{\mathcal{L}_a^{(n-1)}} |\mathcal{F}|^2 \, \mathrm{d}k' \right)^{1/2} = \left(\int_{\varkappa - a}^{\varkappa + a} \mathrm{d}\varkappa \int_{S_{n-2}[\varkappa]} |\widehat{\mathcal{F}}|^2 \, \mathrm{d}\sigma_{n-2}^{(\varkappa)} \right)^{1/2} \leqslant C_2 a^{1/2} \varkappa^{1/q} \|\mathcal{F}\|_{L^p(\mathbb{R}^{n-1})},$$

where $C_2 = C_2(n) > 0$. Therefore,

$$\begin{aligned} \left| \int_{\mathbb{R}^{n-1}} \overline{\breve{u}(x')} \mathcal{F}(x') \, \mathrm{d}x' \right| &= (2\pi)^{n-1} \left| \int_{\mathcal{L}_a^{(n-1)}} \overline{u} \widehat{\mathcal{F}} \, \mathrm{d}k' \right| \\ &\leqslant (2\pi)^{n-1} \|u\|_{L^2(\mathcal{L}_a^{(n-1)})} \|\widehat{\mathcal{F}}\|_{L^2(\mathcal{L}_a^{(n-1)})} \\ &\leqslant (2\pi)^{n-1} C_2 a^{1/2} \varkappa^{1/q} \|u\|_{L^2(\mathcal{L}_a^{(n-1)})} \|\mathcal{F}\|_{L^p(\mathbb{R}^{n-1})} \end{aligned}$$

and

$$\|\breve{u}\|_{L^q(\mathbb{R}^{n-1})} = \sup_{\mathcal{F} \in \mathcal{S}(\mathbb{R}^{n-1}) : \|\mathcal{F}\|_{L^p(\mathbb{R}^{n-1})} = 1} \left| \int_{\mathbb{R}^{n-1}} \overline{\breve{u}(x')} \mathcal{F}(x') \, \mathrm{d}x' \right| \leqslant C_1 a^{1/2} \varkappa^{1/q} \|u\|_{L^2(\mathcal{L}_a^{(n-1)})},$$

where $C_1 = (2\pi)^{n-1} C_2$. □

Let $\mathfrak{L}^{n-1}(e) = \{x \in \mathbb{R}^n : (x, e) = 0\}$. For vectors $x \in \mathbb{R}^n$ we write $x = (x_\parallel, x_\perp)$, where $x_\parallel = (x, e) \in \mathbb{R}$, $x_\perp = x - (x, e)e \in \mathfrak{L}^{n-1}(e)$, $e = |\gamma|^{-1}\gamma$. For functions $\mathcal{F} \in \mathcal{S}(\mathbb{R}^n)$, let us define the norms

$$\|\mathcal{F}\|_{L^2_\parallel L^q_\perp(\mathbb{R}^n)} = \left(\int_{\mathbb{R}} \|\mathcal{F}((x_\parallel, .))\|^2_{L^q(\mathfrak{L}^{n-1}(e))} \, \mathrm{d}x_\parallel \right)^{1/2},$$

$$\|\mathcal{F}\|_{L^\infty_\parallel L^q_\perp(\mathbb{R}^n)} = \operatorname*{ess\,sup}_{x_\parallel \in \mathbb{R}} \|\mathcal{F}((x_\parallel, .))\|_{L^q(\mathfrak{L}^{n-1}(e))}.$$

Denote

$$\widetilde{\mathcal{K}}_a = \{k \in \mathbb{R}^n : |\varkappa - |k_\perp|| \leqslant a, |k_\parallel| \leqslant a\}.$$

For functions $u \in L^2(\widetilde{\mathcal{K}}_a)$, we shall use the notation

$$\widetilde{u}(x_\parallel, k_\perp) = \int_{\mathbb{R}} u(k) \, \mathrm{e}^{\mathrm{i}k_\parallel x_\parallel} \, \mathrm{d}k_\parallel, \qquad x_\parallel \in \mathbb{R}, \quad k \in \mathbb{R}^n.$$

Then

$$\breve{u}(x) = \int_{\mathfrak{L}^{n-1}(e)} \widetilde{u}(x_\parallel, k_\perp) \, \mathrm{e}^{\mathrm{i}(k_\perp, x_\perp)} \, \mathrm{d}k_\perp, \qquad x \in \mathbb{R}^n.$$

Lemma 2.2. *For all functions $u \in L^2(\widetilde{\mathcal{K}}_a)$, the estimate*

$$\|\breve{u}\|_{L^q(\mathbb{R}^n)} \leqslant C_3 a^{1/2+1/n} \varkappa^{1/2-1/n} \|u\|_{L^2(\widetilde{\mathcal{K}}_a)}$$

is valid, where $C_3 = C_3(n) > 0$.

Proof. From lemma 2.1 it follows that

$$\|\breve{u}((x_\parallel, .))\|_{L^q(\mathfrak{L}^{n-1}(e))} \leqslant C_1' \|\widetilde{u}(x_\parallel, .)\|_{L^2(\mathfrak{L}^{n-1}(e))}$$

for all $x_\parallel \in \mathbb{R}$, where $C_1' = C_1 a^{1/2}\varkappa^{1/q}$. Therefore the following estimates hold:

$$\begin{aligned}\|\check{u}\|_{L^2_\parallel L^q_\perp(\mathbb{R}^n)} &= \left(\int_{\mathbb{R}} \|\check{u}((x_\parallel,.))\|^2_{L^q(\mathfrak{L}^{n-1}(e))}\,\mathrm{d}x_\parallel\right)^{1/2}\\ &\leqslant C_1'\left(\int_{\mathbb{R}} \|\widetilde{u}(x_\parallel,.)\|^2_{L^2(\mathfrak{L}^{n-1}(e))}\,\mathrm{d}x_\parallel\right)^{1/2} = C_1'\left(\int_{\mathfrak{L}^{n-1}(e)}\int_{\mathbb{R}} |\widetilde{u}(x_\parallel,k_\perp)|^2\,\mathrm{d}k_\perp\,\mathrm{d}x_\parallel\right)^{1/2}\\ &= \frac{C_1'}{\sqrt{2\pi}}\left(\int_{\mathfrak{L}^{n-1}(e)}\int_{\mathbb{R}} |u(k)|^2\,\mathrm{d}k_\perp\,\mathrm{d}k_\parallel\right)^{1/2} = \frac{C_1'}{\sqrt{2\pi}}\|u\|_{L^2(\widetilde{\mathcal{K}}_a)}, \qquad (19)\end{aligned}$$

$$\begin{aligned}\|\check{u}\|_{L^\infty_\parallel L^q_\perp(\mathbb{R}^n)} &= \operatorname*{ess\,sup}_{x_\parallel\in\mathbb{R}} \|\check{u}((x_\parallel,.))\|_{L^q(\mathfrak{L}^{n-1}(e))} \leqslant C_1'\operatorname*{ess\,sup}_{x_\parallel\in\mathbb{R}} \|\widetilde{u}(x_\parallel,.)\|_{L^2(\mathfrak{L}^{n-1}(e))}\\ &= C_1'\operatorname*{ess\,sup}_{x_\parallel\in\mathbb{R}}\left(\int_{\mathfrak{L}^{n-1}(e)}\left|\int_{-a}^{a} u(k)\,\mathrm{e}^{\mathrm{i}k_\parallel x_\parallel}\,\mathrm{d}k_\parallel\right|^2\mathrm{d}k_\perp\right)^{1/2}\\ &\leqslant C_1'(2a)^{1/2}\left(\int_{\mathfrak{L}^{n-1}(e)}\left(\int_{\mathbb{R}}|u(k)|^2\,\mathrm{d}k_\parallel\right)\mathrm{d}k_\perp\right)^{1/2} = C_1'(2a)^{1/2}\|u\|_{L^2(\widetilde{\mathcal{K}}_a)}. \qquad (20)\end{aligned}$$

Since the inequality

$$\|f\|_{L^q(\mathbb{R})} \leqslant \|f\|^{2/q}_{L^2(\mathbb{R})}\|f\|^{1-2/q}_{L^\infty(\mathbb{R})}$$

is valid for all functions $f \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$, from (19) and (20) we obtain the estimate

$$\|\check{u}\|_{L^q(\mathbb{R}^n)} \leqslant \left(\|\check{u}\|_{L^2_\parallel L^q_\perp(\mathbb{R}^n)}\right)^{2/q}\left(\|\check{u}\|_{L^\infty_\parallel L^q_\perp(\mathbb{R}^n)}\right)^{1-2/q} \leqslant C_3 a^{1/2+1/n}\varkappa^{1/2-1/n}\|u\|_{L^2(\widetilde{\mathcal{K}}_a)},$$

where $C_3 = C_3(n) = C_1 2^{1/n}(2\pi)^{-1/2+1/n}$. □

For a fixed vector $k \in \mathbb{R}^n$ $\left(\text{and for } 0 < a \leqslant \frac{3}{4}\varkappa\right)$, define the sets

$$\mathcal{K}_a = \{N \in \Lambda^* : k + 2\pi N \in \widetilde{\mathcal{K}}_a\}.$$

Let $\operatorname{diam} K^*$ be diameter of the fundamental domain K^*. For any set $\mathcal{C} \subseteq \Lambda^*$, let us denote $\mathcal{H}(\mathcal{C}) = \{\phi \in L^2(K) : \phi_N = 0 \text{ for } N \in \Lambda^*\backslash\mathcal{C}\}$, $\mathcal{H}(\emptyset) = \{0\}$, $\mathcal{H}(\Lambda^*) = L^2(K)$.

Lemma 2.3. *Let* $\varkappa \geqslant 4\pi \operatorname{diam} K^*$ *and let* $\pi \operatorname{diam} K^* \leqslant a \leqslant \varkappa/2$. *Then for any function* $\mathcal{F} \in \mathcal{H}(\mathcal{K}_a)$ *the inequality*

$$\|\mathcal{F}\|_{L^q(K)} \leqslant C_4 a^{1/2+1/n}\varkappa^{1/2-1/n}\|\mathcal{F}\|_{L^2(K)} \qquad (21)$$

holds, where $C_4 = C_4(n) > 0$.

Proof. Denote by $\widehat{\mathcal{L}}$ the linear transformation on the space $\mathbb{R}^n$ such that $\widehat{\mathcal{L}}E_j = \mathcal{E}_j$, $j = 1,\ldots,n$ (where $\{\mathcal{E}_j\}$ is the fixed orthogonal basis in $\mathbb{R}^n$). Then also $(\widehat{\mathcal{L}}^{-1})^*E_l^* = \mathcal{E}_l$, $l = 1,\ldots,n$, and $|\det\widehat{\mathcal{L}}| = v^{-1}(K) = v(K^*)$ (here $\{E_j\}$ and $\{E_j^*\}$ are the bases in the lattices Λ and Λ^*, respectively, $(E_j^*, E_l) = \delta_{jl}$). Let Ξ be the set of functions $\omega \in \mathcal{S}(\mathbb{R}^n)$ such that $\widehat{\omega} \in C_0^\infty(\mathbb{R}^n)$, $\widehat{\omega}(\widetilde{k}) \geqslant 0$ for all $\widetilde{k} \in \mathbb{R}^n$, $\widehat{\omega}(\widetilde{k}) = 0$ if $|\widetilde{k}_j| \geqslant \frac{1}{2}$ for some index $j \in \{1,\ldots,n\}$, and

$$\int_{\mathbb{R}^n}\widehat{\omega}^2(\widetilde{k})\,\mathrm{d}\widetilde{k} = (2\pi)^n\int_{\mathbb{R}^n}|\omega(x)|^2\,\mathrm{d}x = 1.$$

For functions $\omega \in \Xi$, we define the functions $\Omega(x) = \omega(2\pi\widehat{\mathcal{L}}x)$, $x \in \mathbb{R}^n$. One has

$$\widehat{\Omega}(\widetilde{k}) = \frac{v(K)}{(2\pi)^n}\widehat{\omega}\left(\frac{1}{2\pi}(\widehat{\mathcal{L}}^{-1})^*\widetilde{k}\right), \qquad \widetilde{k} \in \mathbb{R}^n.$$

Consequently,

$$\int_{\mathbb{R}^n} \widehat{\Omega}^2(\widetilde{k})\,\mathrm{d}\widetilde{k} = \frac{v(K)}{(2\pi)^n} \tag{22}$$

and

$$\widehat{\Omega}(\widetilde{k})\widehat{\Omega}(\widetilde{k}-2\pi N) \equiv 0, \qquad \widetilde{k} \in \mathbb{R}^n, \tag{23}$$

for all $N \in \Lambda^*\backslash\{0\}$. We write $b = \pi \operatorname{diam} K^*$. The estimate $a + b \leqslant \frac{3}{4}\varkappa$ holds. Since

$$\widehat{\Omega\mathcal{F}}(\widetilde{k}) = \sum_{N\in\mathcal{K}_a} \mathcal{F}_N \widehat{\Omega}(\widetilde{k}-2\pi N), \qquad \widetilde{k} \in \mathbb{R}^n,$$

the equality $\widehat{\Omega\mathcal{F}}(\widetilde{k}) = 0$ is fulfilled in the case where $\widetilde{k} - k \in \mathbb{R}^n\backslash\widetilde{\mathcal{K}}_{a+b}$. Hence, by lemma 2.2,

$$\|\Omega\mathcal{F}\|_{L^q(\mathbb{R}^n)} \leqslant C_3 a^{1/2+1/n}\varkappa^{1/2-1/n}\|\widehat{\Omega\mathcal{F}}\|_{L^2(k+\widetilde{\mathcal{K}}_{a+b})}. \tag{24}$$

Furthermore (see (22), (23)),

$$\begin{aligned}\|\widehat{\Omega\mathcal{F}}\|_{L^2(k+\widetilde{\mathcal{K}}_{a+b})} &= \int_{\mathbb{R}^n}\left|\sum_{N\in\mathcal{K}_a}\mathcal{F}_N\widehat{\Omega}(\widetilde{k}-2\pi N)\right|^2 \mathrm{d}\widetilde{k} \\ &= \left(\int_{\mathbb{R}^n}\widehat{\Omega}^2(\widetilde{k})\,\mathrm{d}\widetilde{k}\right)\sum_{N\in\mathcal{K}_a}|\mathcal{F}_N|^2 = (2\pi)^{-n}\|\mathcal{F}\|_{L^2(K)}. \end{aligned} \tag{25}$$

On the other hand,

$$\|\Omega\mathcal{F}\|_{L^q(K)} \leqslant \|\Omega\mathcal{F}\|_{L^q(\mathbb{R}^n)} \tag{26}$$

and since one can pick an arbitrary function $\omega \in \Xi$, it is not hard to obtain the estimate

$$\|\mathcal{F}\|_q \doteq \|\mathcal{F}\|_{L^q(K)} \leqslant C_5 \sup_{\omega\in\Xi}\|\Omega\mathcal{F}\|_{L^q(K)}, \tag{27}$$

where $C_5 = C_5(n) > 0$. Finally, estimate (21) with the constant $C_4 = C_3C_5$ follows from (24), (25), (26) and (27). □

Lemma 2.4. *Let* $\varkappa \geqslant 4\pi \operatorname{diam} K^*$ *and let* $\pi \operatorname{diam} K^* \leqslant a \leqslant \varkappa/2$*. Then for any* $\varepsilon > 0$ *there is a constant* $C(n,\varepsilon) > 0$ *such that for all functions* $\mathcal{W} \in L^n_w(K)$ *and* $\phi \in \mathcal{H}(\mathcal{K}_a)$ *the inequality*

$$\|\mathcal{W}\phi\| \leqslant C(n,\varepsilon)a^{1/2+1/n}\varkappa^{1/2-1/n}\left(\frac{\varkappa}{a}\right)^{\varepsilon}\|\mathcal{W}\|_{n,w}\|\phi\| \tag{28}$$

holds.

Proof. We may assume that $\varepsilon < \min\left\{\frac{n-2}{8}, \frac{1}{4}\right\}$. Define the numbers $\varepsilon_1 = 8\varepsilon/(n-2) \in (0,1)$, $\varepsilon_2 = 4\varepsilon \in (0,1)$, and let $\phi \in \mathcal{H}(\mathcal{K}_a)$. For functions $\mathcal{V}_1 \in L^2(K)$ and $\mathcal{V}_2 \in L^\infty(K)$, the following estimates are valid:

$$\begin{aligned}\|\mathcal{V}_1\phi\| &\leqslant \|\mathcal{V}_1\|_2\|\phi\|_\infty \leqslant \|\mathcal{V}_1\|_2\left(\sum_{N\in\mathcal{K}_a}|\phi_N|\right) \\ &\leqslant \|\mathcal{V}_1\|_2\left(\sum_{N\in\mathcal{K}_a}1\right)^{1/2}\left(\sum_{N\in\mathcal{K}_a}|\phi_N|^2\right)^{1/2} \leqslant C_6 a\varkappa^{(n-2)/2}\|\mathcal{V}_1\|_2\|\phi\|, \end{aligned} \tag{29}$$

where $C_6 = C_6(n) > 0$, and

$$\|\mathcal{V}_2\phi\| \leqslant \|\mathcal{V}_2\|_\infty\|\phi\| \tag{30}$$

(here $\|\cdot\| = \|\cdot\|_2 \doteq \|\cdot\|_{L^2(K)}$). On the other hand, using lemma 2.3, for functions $\mathcal{V} \in L^n(K)$, we derive

$$\|\mathcal{V}\phi\| \leqslant \|\mathcal{V}\|_n\|\phi\|_q \leqslant C_4 a^{1/2+1/n}\varkappa^{1/2-1/n}\|\mathcal{V}\|_n\|\phi\|. \tag{31}$$

Now, pick the numbers $n_1 \in (2, n)$ and $n_2 \in (n, +\infty)$ such that

$$\frac{1}{n_1} = \frac{\varepsilon_1}{2} + \frac{1-\varepsilon_1}{n}, \qquad \frac{1}{n_2} = \frac{1-\varepsilon_2}{n}.$$

For functions $\mathcal{W}_j \in L^{n_j}(K)$, $j = 1, 2$, from estimates (29) and (31) for $j = 1$, and estimates (30) and (31) for $j = 2$, with the help of interpolation (expressing functions $\mathcal{W}_j$ as sums of 'large' and 'small' ones (see, e.g., [54, 55])), we obtain

$$\|\mathcal{W}_1\phi\| \leqslant 2(C_6 a \varkappa^{(n-2)/2})^{\varepsilon_1} (C_4 a^{1/2+1/n} \varkappa^{1/2-1/n})^{1-\varepsilon_1} \|\mathcal{W}_1\|_{n_1} \|\phi\|, \tag{32}$$

$$\|\mathcal{W}_2\phi\| \leqslant 2(C_4 a^{1/2+1/n} \varkappa^{1/2-1/n})^{1-\varepsilon_2} \|\mathcal{W}_2\|_{n_2} \|\phi\|, \tag{33}$$

respectively. Again applying the interpolation (expressing functions $\mathcal{W} \in L^n_w(K)$ as sums of 'large' functions $\mathcal{W}_1 \in L^{n_1}(K)$ and 'small' functions $\mathcal{W}_2 \in L^{n_2}(K)$ (see [54, 55] and also [56])), from (32) and (33), we derive estimate (28) with some constant $C(n, \varepsilon) > 0$. □

Define the operators

$$\widehat{G}_\pm \phi = \widehat{G}_\pm(k + \mathrm{i}\varkappa e)\phi = \sum_{N \in \Lambda^*} G^\pm_N(k + \mathrm{i}\varkappa e)\phi_N \,\mathrm{e}^{2\pi \mathrm{i}(N,x)},$$
$$\phi \in D(\widehat{G}_\pm) = \widetilde{H}^1(K) \subset L^2(K).$$

We have $\widehat{L} = \widehat{G}_+\widehat{G}_-$. Since the vector $k \in \mathbb{R}^n$ is assumed to satisfy the condition $|(k, \gamma)| = \pi$, we get $G^+_N(k + \mathrm{i}\varkappa e) \geqslant G^-_N(k + \mathrm{i}\varkappa e) \geqslant \pi|\gamma|^{-1}$ for all $\varkappa \geqslant 0$ and all $N \in \Lambda^*$. Hence for all $\zeta \in \mathbb{C}$, we can also define the operators

$$\widehat{G}^\zeta_\pm \phi = \widehat{G}^\zeta_\pm(k + \mathrm{i}\varkappa e)\phi = \sum_{N \in \Lambda^*} \left(G^\pm_N(k + \mathrm{i}\varkappa e)\right)^\zeta \phi_N \,\mathrm{e}^{2\pi \mathrm{i}(N,x)},$$
$$\phi \in D(\widehat{G}^\zeta_\pm) = \begin{cases} \widetilde{H}^{\mathrm{Re}\,\zeta}(K) & \text{if} \quad \mathrm{Re}\,\zeta > 0, \\ L^2(K) & \text{if} \quad \mathrm{Re}\,\zeta \leqslant 0. \end{cases}$$

Given $\varkappa \geqslant \max\{8, 4\pi \operatorname{diam} K^*\}$, we choose the numbers $h \in [2, 4)$ and $l \in \mathbb{N}\backslash\{1\}$ such that $h^l = \varkappa/2$. Let $m \in \mathbb{N}$ be the smallest number for which $h^m \geqslant \pi \operatorname{diam} K^*$ (then $m < l$). Denote

$$\mathcal{K}(m) = \{N \in \Lambda^* : G^-_N(k + \mathrm{i}\varkappa e) \leqslant h^m\},$$
$$\mathcal{K}(j) = \{N \in \Lambda^* : h^{j-1} < G^-_N(k + \mathrm{i}\varkappa e) \leqslant h^j\}, \qquad j = m+1, \ldots, l,$$
$$\mathcal{K} = \bigcup_{j=m}^{l} \mathcal{K}(j); \qquad \mathcal{K} \subseteq \mathcal{K}_{\varkappa/2}.$$

The following estimates are valid:

$$\sqrt{\frac{\pi}{|\gamma|}}\|\phi\| \leqslant \left\|\widehat{G}^{1/2}_-\phi\right\|, \qquad \phi \in \mathcal{H}(\mathcal{K}(m)), \tag{34}$$

$$h^{(j-1)/2}\|\phi\| \leqslant \left\|\widehat{G}^{1/2}_-\phi\right\|, \qquad \phi \in \mathcal{H}(\mathcal{K}(j)), \qquad j = m+1, \ldots, l. \tag{35}$$

For functions $\phi \in \mathcal{H}(\mathcal{K})$, define the functions

$$\phi_j = \sum_{N \in \mathcal{K}(j)} \phi_N \,\mathrm{e}^{2\pi \mathrm{i}(N,x)}, \qquad j = m, \ldots, l.$$

We have $\phi_j \in \mathcal{H}(\mathcal{K}(j))$, $j = m, \ldots, l$, and $\phi = \sum_{j=m}^{l} \phi_j$.

Using lemma 2.4 and estimates (34) and (35), for all $\varepsilon \in \left(0, \frac{1}{n}\right)$, we deduce that

$$\|\mathcal{W}\phi\| \leqslant \sum_{j=m}^{l} \|\mathcal{W}\phi_j\| \leqslant C(n,\varepsilon)\|\mathcal{W}\|_{n,w}\varkappa^{1/2-1/n-\varepsilon} \sum_{j=m}^{l} h^{j(1/2+1/n-\varepsilon)}\|\phi_j\|$$
$$\leqslant C(n,\varepsilon)\|\mathcal{W}\|_{n,w}\varkappa^{1/2-1/n-\varepsilon} \times \left(\sqrt{\frac{\pi}{|\gamma|}} h^{m(1/2+1/n-\varepsilon)} \big\|\widehat{G}_-^{1/2}\phi_m\big\| \right.$$
$$\left. + \sum_{j=m+1}^{l} h^{-j/2+1/2} h^{j(1/2+1/n-\varepsilon)} \big\|\widehat{G}_-^{1/2}\phi_j\big\| \right)$$
$$\leqslant C(n,\varepsilon)\|\mathcal{W}\|_{n,w}\varkappa^{1/2} \times \left(\sqrt{\frac{\pi}{|\gamma|}} h^{m(1/2+1/n-\varepsilon)} \varkappa^{-1/n+\varepsilon} \big\|\widehat{G}_-^{1/2}\phi_m\big\| \right.$$
$$\left. + 2^{-1/n+\varepsilon} h^{1/2} \sum_{j_1=0}^{l-m-1} h^{-j_1(1/n-\varepsilon)} \big\|\widehat{G}_-^{1/2}\phi_j\big\| \right). \tag{36}$$

Now, let $\varepsilon = \frac{1}{2n}$. Then (36) implies that there is a number $\varkappa_0 > 0$ such that for all $\varkappa \geqslant \varkappa_0$, all vectors $k \in \mathbb{R}^n$ with $|(k,\gamma)| = \pi$, and all functions $\phi \in \mathcal{H}(\mathcal{K})$, the inequality

$$\|\mathcal{W}\phi\| \leqslant \widetilde{C}_1 \|\mathcal{W}\|_{n,w}\varkappa^{1/2} \big\|\widehat{G}_-^{1/2}\phi\big\| \leqslant \widetilde{C}_1 \|\mathcal{W}\|_{n,w} \|\widehat{L}^{1/2}(k + \mathrm{i}\varkappa e)\phi\| \tag{37}$$

holds, where $\widetilde{C}_1 = \widetilde{C}_1(n) = 4C\left(n, \frac{1}{2n}\right)(1 - 2^{-1/(2n)})^{-1}$. On the other hand, for all $\phi \in \widetilde{H}^1(K)$ and all $k \in \mathbb{R}^n$, we have

$$\|\mathcal{W}\phi\| \leqslant \|\mathcal{W}\|_{n,w}\left(\widetilde{C}_2\left(\sum_{j=1}^{n}\left\|\left(k_j - \mathrm{i}\frac{\partial}{\partial x_j}\right)\phi\right\|^2\right)^{1/2} + \widetilde{C}_3\|\phi\|\right), \tag{38}$$

where $\widetilde{C}_2 = \widetilde{C}_2(n) > 0$ and $\widetilde{C}_3 = \widetilde{C}_3(n,\Lambda) > 0$ (see [56] and also [35, 34]). Since $G_N^-(k + \mathrm{i}\varkappa e) \geqslant \frac{1}{3}|k + 2\pi N|$ and, consequently, $G_N^+G_N^- \geqslant \frac{1}{3}|k + 2\pi N|^2$ for all $N \in \Lambda^*\backslash\mathcal{K}$, from (38) it follows that there exists a number $\widetilde{\varkappa}_0 > 0$ such that for all $\varkappa \geqslant \widetilde{\varkappa}_0$, all vectors $k \in \mathbb{R}^n$ with $|(k,\gamma)| = \pi$, and all functions $\phi \in \widetilde{H}^1(K) \cap \mathcal{H}(\Lambda^*\backslash\mathcal{K})$, the inequality

$$\|\mathcal{W}\phi\| \leqslant 2\widetilde{C}_2\|\mathcal{W}\|_{n,w}\|\widehat{L}^{1/2}(k + \mathrm{i}\varkappa e)\phi\| \tag{39}$$

is valid. Now, theorem 1.2 directly follows from (37) and (39).

3. Proof of theorem 1.3

Without loss of generality we shall assume that $A_0 = 0$.

Let $\mathcal{F}$ be a nonnegative function from the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \mathcal{F}(x)\,\mathrm{d}x = 1$ and the Fourier transform $\widehat{\mathcal{F}}$ has a compact support; $\mathcal{F}_r(x) = r^n\mathcal{F}(rx)$, $r > 0$, $x \in \mathbb{R}^n$. For $r > 0$, we use the notation

$$A^{(0)}(x) = \int_{\mathbb{R}^n} A(x - y)\mathcal{F}_r(y)\,\mathrm{d}y, \qquad x \in \mathbb{R}^n.$$

The function $A^{(0)} : \mathbb{R}^n \to \mathbb{R}^n$ is a trigonometric polynomial with the period lattice $\Lambda \subset \mathbb{R}^n$, $A_0^{(0)} = 0$. Furthermore, the function $A^{(0)}$ obeys the condition (A_2) of theorem 0.1 and, moreover,

$$|||A^{(0)}|||_{2,\gamma} \leqslant |||A|||_{2,\gamma} \leqslant \mathfrak{a}, \qquad \theta(\Lambda,\gamma,\mu,h;A^{(0)}) \leqslant \theta(\Lambda,\gamma,\mu,h;A) \leqslant \Theta.$$

For any $\varepsilon > 0$, taking the number $r > 0$ to be sufficiently large, we can also suppose that for the function $A^{(1)} \doteq A - A^{(0)}$, the estimate

$$|||A^{(1)}|||_{2,\gamma} \leqslant \varepsilon |||A|||_{2,\gamma} \tag{40}$$

holds (see, e.g., [45, 46]). Besides, the condition (A_1) is fulfilled for the function $A^{(1)}$ (and $A_0^{(1)} = 0$).

Since the condition (A_1) implies inequalities (4) for the functions A and $A^{(1)}$, we get

$$\begin{aligned} W(A; k + \mathrm{i}\varkappa e; \psi, \phi) &= W(A^{(0)}; k + \mathrm{i}\varkappa e; \psi, \phi) + 2\sum_{j=1}^{n} \big(A_j^{(0)}\psi, A_j^{(1)}\phi\big) \\ &\quad - \sum_{j=1}^{n} \left(A_j^{(1)}\psi, \left(-\mathrm{i}\frac{\partial}{\partial x_j} + k_j + \mathrm{i}\varkappa e_j\right)\phi\right) \\ &\quad - \sum_{j=1}^{n} \left(\left(-\mathrm{i}\frac{\partial}{\partial x_j} + k_j - \mathrm{i}\varkappa e_j\right)\psi, A_j^{(1)}\phi\right) \\ &\quad + \sum_{j=1}^{n} (A_j^{(1)}\psi, A_j^{(1)}\phi), \qquad \psi, \phi \in \widetilde{H}^1(K) \end{aligned} \tag{41}$$

for all $k \in \mathbb{R}^n$ and all $\varkappa \geqslant 0$. For any measurable function $e^* : K \to S_{n-1}$ and for all $\varkappa \geqslant 0$, all $k \in \mathbb{R}^n$, and all $\phi \in \widetilde{H}^1(K)$

$$\begin{aligned} \left\| \sum_{j=1}^{n} e_j^*\left(k_j - \mathrm{i}\frac{\partial}{\partial x_j}\right)\phi \right\|^2 &= \int_K \left| \sum_{j=1}^{n} e_j^*\left(k_j - \mathrm{i}\frac{\partial}{\partial x_j}\right)\phi \right|^2 \mathrm{d}x \\ &\leqslant \int_K \sum_{j=1}^{n} \left| \left(k_j - \mathrm{i}\frac{\partial}{\partial x_j}\right)\phi \right|^2 \mathrm{d}x = v(K) \sum_{N\in\Lambda^*} |k + 2\pi N|^2 |\phi_N|^2 \\ &\leqslant \|\widehat{G}_+(k + \mathrm{i}\varkappa e)\phi\|^2. \end{aligned} \tag{42}$$

On the other hand, from lemma 1.1 it follows that for all $\varkappa \geqslant 0$, all vectors $k \in \mathbb{R}^n$ with $|(k, \gamma)| = \pi$, and all functions $\phi \in \widetilde{H}^1(K)$

$$\| |A^{(1)}|\phi \| \leqslant C_2 \| |A^{(1)}| \|_{2,\gamma} \left\| \left(k_1 - \mathrm{i}\frac{\partial}{\partial x_1}\right)\phi \right\| \leqslant C_2 \| |A^{(1)}| \|_{2,\gamma} \|\widehat{G}_-(k + \mathrm{i}\varkappa e)\phi\|, \tag{43}$$

where $C_2 = C_2(n, |\gamma|) > 0$. Given $\phi \in \widetilde{H}^1(K)$, let us define the functions

$$\phi^{(0)}(x) \doteq \sum_{N\in\Lambda^*:2\pi|N|\leqslant 2\varkappa} \phi_N\, \mathrm{e}^{2\pi \mathrm{i}(N,x)}, \qquad \phi^{(1)}(x) \doteq \phi(x) - \phi^{(0)}(x), \qquad x \in \mathbb{R}^n.$$

Since $G_N^-(k + \mathrm{i}\varkappa e) > \frac{1}{3}G_N^+(k + \mathrm{i}\varkappa e)$ for all $N \in \Lambda^*$ with $2\pi|N| > 2\varkappa$, from (42) (where we put $e^*(x) = |A(x)|^{-1}A(x)$ if $A(x) \neq 0, x \in K$) and (43) (under the condition $|(k, \gamma)| = \pi$) we derive

$$\begin{aligned} \left| \sum_{j=1}^{n} \left(A_j^{(1)}\psi, \left(k_j - \mathrm{i}\frac{\partial}{\partial x_j}\right)\phi^{(1)}\right) \right| &\leqslant C_2 \| |A^{(1)}| \|_{2,\gamma} \|\widehat{G}_-\psi\| \cdot \|\widehat{G}_+\phi^{(1)}\| \\ &\leqslant \sqrt{3} C_2 \| |A^{(1)}| \|_{2,\gamma} \|\widehat{L}^{1/2}\psi\| \cdot \|\widehat{L}^{1/2}\phi^{(1)}\|, \qquad \psi, \phi \in \widetilde{H}^1(K). \end{aligned} \tag{44}$$

Lemma 3.1. *For all $\varkappa \geqslant 0$, all vectors $k \in \mathbb{R}^n$ with $|(k, \gamma)| = \pi$, and all functions $\psi, \phi \in \widetilde{H}^1(K)$, the estimates*

$$\big|(A_j^{(1)}\psi, \phi)\big| \leqslant C_2 \| |A^{(1)}| \|_{2,\gamma} \big\|\widehat{G}_-^{1/2}(k + \mathrm{i}\varkappa e)\psi\big\| \cdot \|\widehat{G}_-^{1/2}(k + \mathrm{i}\varkappa e)\phi\|, \qquad j = 1, \ldots, n, \tag{45}$$

hold, where $C_2 = C_2(n, |\gamma|)$ is the constant from (43).

Proof. For all $\zeta \in \mathbb{C}$ with $0 \leqslant \mathrm{Re}\,\zeta \leqslant 1$ (and for fixed $\varkappa \geqslant 0$ and k), define the operators

$$\widehat{\mathcal{R}}_j(\zeta) = \widehat{G}_-^{-1+\zeta} A_j^{(1)} \widehat{G}_-^{-\zeta}, \qquad j = 1, \ldots, n,$$

$D(\widehat{\mathcal{R}}_j(\zeta)) = \widetilde{H}^1(K) \subset L^2(K)$. From (43) it follows that for all functions $\psi, \phi \in \widetilde{H}^1(K)$, the functions $\mathbb{C} \ni \zeta \to (\psi, \widehat{\mathcal{R}}_j(\zeta)\phi)$ are uniformly bounded for $0 \leqslant \mathrm{Re}\,\zeta \leqslant 1$ and analytic for $0 < \mathrm{Re}\,\zeta < 1$. Furthermore,

$$|(\psi, \widehat{\mathcal{R}}_j(\zeta)\phi)| \leqslant C_2 \| |A^{(1)}| \|_{2,\gamma} \|\psi\| \cdot \|\phi\| \tag{46}$$

if $\mathrm{Re}\,\zeta = 0$ or $\mathrm{Re}\,\zeta = 1$. Hence estimates (46) hold for all $\zeta \in \mathbb{C}$ with $0 \leqslant \mathrm{Re}\,\zeta \leqslant 1$. In particular, for $\zeta = \frac{1}{2}$, inequalities (46) yield the inequalities

$$\left\| \left(\widehat{G}_-^{-1/2}\psi, A_j^{(1)}\widehat{G}_-^{-1/2}\phi\right) \right\| \leqslant C_2 \| |A^{(1)}| \|_{2,\gamma} \|\psi\| \cdot \|\phi\|$$

which imply inequalities (45) for functions $\psi, \phi \in \widetilde{H}^{3/2}(K)$. Since the set $\widetilde{H}^{3/2}(K)$ is dense in the Sobolev class $\widetilde{H}^1(K)$, by continuity, estimates (45) are also valid for all functions $\psi, \phi \in \widetilde{H}^1(K)$. □

By lemma 3.1, it follows that

$$\begin{aligned}
\left| \sum_{j=1}^n \left(A_j^{(1)}\psi, \left(k_j - \mathrm{i}\frac{\partial}{\partial x_j}\right)\phi^{(0)} \right) \right| &\leqslant C_2 \| |A^{(1)}| \|_{2,\gamma} \left\| \widehat{G}_-^{1/2}\psi \right\| \cdot \sum_{j=1}^n \left\| \widehat{G}_-^{1/2}\left(k_j - \mathrm{i}\frac{\partial}{\partial x_j}\right)\phi^{(0)} \right\| \\
&\leqslant 2nC_2\varkappa \| |A^{(1)}| \|_{2,\gamma} \left\| \widehat{G}_-^{1/2}\psi \right\| \cdot \left\| \widehat{G}_-^{1/2}\phi^{(0)} \right\| \\
&\leqslant 2nC_2 \| |A^{(1)}| \|_{2,\gamma} \|\widehat{L}^{1/2}\psi\| \cdot \|\widehat{L}^{1/2}\phi^{(0)}\|.
\end{aligned}$$

This inequality and inequality (44) imply that for all $\varkappa \geqslant 0$, all vectors $k \in \mathbb{R}^n$ with $|(k, \gamma)| = \pi$, and all functions $\psi, \phi \in \widetilde{H}^1(K)$, the following estimate holds:

$$\begin{aligned}
&\left| \sum_{j=1}^n \left(A_j^{(1)}\psi, \left(k_j - \mathrm{i}\frac{\partial}{\partial x_j}\right)\phi \right) \right| \\
&\qquad \leqslant (\sqrt{3} + 2n) C_2 \| |A^{(1)}| \|_{2,\gamma} \|\widehat{L}^{1/2}(k + \mathrm{i}\varkappa e)\psi\| \cdot \|\widehat{L}^{1/2}(k + \mathrm{i}\varkappa e)\phi\|.
\end{aligned} \tag{47}$$

By analogy with lemma 3.1, using (43), we obtain

$$\left| \sum_{j=1}^n \left(A_j^{(1)}\psi, (\mathrm{i}\varkappa e_j)\phi \right) \right| = \left| \sum_{j=1}^n \left((-\mathrm{i}\varkappa e_j)\psi, A_j^{(1)}\phi \right) \right| \leqslant C_2 \| |A^{(1)}| \|_{2,\gamma} \|\widehat{L}^{1/2}\psi\| \cdot \|\widehat{L}^{1/2}\phi\|$$

(for all functions $\psi, \phi \in \widetilde{H}^1(K)$). The last inequality and (47) yield

$$\begin{aligned}
&\left| \sum_{j=1}^n \left(A_j^{(1)}\psi, \left(-\mathrm{i}\frac{\partial}{\partial x_j} + k_j + \mathrm{i}\varkappa e_j \right)\phi \right) + \sum_{j=1}^n \left(\left(-\mathrm{i}\frac{\partial}{\partial x_j} + k_j - \mathrm{i}\varkappa e_j \right)\psi, A_j^{(1)}\phi \right) \right| \\
&\qquad \leqslant 2(1 + \sqrt{3} + 2n) C_2 \| |A^{(1)}| \|_{2,\gamma} \|\widehat{L}^{1/2}\psi\| \cdot \|\widehat{L}^{1/2}\phi\|.
\end{aligned} \tag{48}$$

We also have (see (43))

$$\begin{aligned}
\left| \sum_{j=1}^n \left(A_j^{(0)}\psi, A_j^{(1)}\phi \right) \right| &\leqslant \| |A^{(0)}|\psi\| \cdot \| |A^{(1)}|\phi\| \\
&\leqslant C_2^2 \| |A| \|_{2,\gamma} \| |A^{(1)}| \|_{2,\gamma} \|\widehat{G}_-\psi\| \cdot \|\widehat{G}_-\phi\| \\
&\leqslant C_2^2 \| |A| \|_{2,\gamma} \| |A^{(1)}| \|_{2,\gamma} \|\widehat{L}^{1/2}\psi\| \cdot \|\widehat{L}^{1/2}\phi\|,
\end{aligned} \tag{49}$$

$$\begin{aligned}
\left| \sum_{j=1}^n \left(A_j^{(1)}\psi, A_j^{(1)}\phi \right) \right| &\leqslant \| |A^{(1)}|\psi\| \cdot \| |A^{(1)}|\phi\| \\
&\leqslant C_2^2 \| |A^{(1)}| \|_{2,\gamma}^2 \|\widehat{G}_-\psi\| \cdot \|\widehat{G}_-\phi\| \\
&\leqslant C_2^2 \| |A^{(1)}| \|_{2,\gamma}^2 \|\widehat{L}^{1/2}\psi\| \cdot \|\widehat{L}^{1/2}\phi\|, \qquad \psi, \phi \in \widetilde{H}^1(K).
\end{aligned} \tag{50}$$

Since the number $\varepsilon > 0$ can be chosen arbitrarily small in the condition (40), we get from (41) and (48), (49) and (50) that it suffices to prove theorem 1.3 only for the function $A^{(0)}$. Indeed, it suffices to assume that the number $\varepsilon > 0$ obeys the condition

$$2\varepsilon(1+\sqrt{3}+2n)C_2\mathfrak{a}+\varepsilon(\varepsilon+2)C_2^2\mathfrak{a}^2 < \tfrac{1}{2}C_1$$

and then replace $\frac{1}{2}C_1$ by C_1. Therefore, in what follows, using the former notation $A^{(0)} = A$ we shall suppose that the magnetic potential A is a trigonometric polynomial.

Let $\widehat{\alpha}_j$, $j = 1, \ldots, n$, be Hermitian $M \times M$-matrices such that

$$\widehat{\alpha}_j\widehat{\alpha}_l + \widehat{\alpha}_l\widehat{\alpha}_j = 2\delta_{jl}\widehat{I}_M, \tag{51}$$

where $\widehat{I}_M$ is the identity $M \times M$-matrix and δ_{jl} is the Kronecker delta. Such matrices exist for $M = \frac{n+1}{2}$ if $n \in 2\mathbb{N}+1$, and for $M = \frac{n}{2}+1$ if $n \in 2\mathbb{N}$. Let

$$\widehat{\mathcal{D}}(A; k+\mathrm{i}\varkappa e) = \sum_{j=1}^{n}\widehat{\alpha}_j\left(-\mathrm{i}\frac{\partial}{\partial x_j} - A_j + k_j + \mathrm{i}\varkappa e_j\right)$$

be the Dirac operator acting on $L^2(K, \mathbb{C}^M)$ with the domain $D(\widehat{\mathcal{D}}(A; k+\mathrm{i}\varkappa e)) = \widetilde{H}^1(K; \mathbb{C}^M)$, $k \in \mathbb{R}^n$, $\varkappa \geqslant 0$. We have

$$\begin{aligned}\widehat{\mathcal{D}}^2(A; k+\mathrm{i}\varkappa e) &= \widehat{H}(A; k+\mathrm{i}\varkappa e)\otimes\widehat{I}_M + \frac{\mathrm{i}}{2}\sum_{j\neq l}\left(\frac{\partial A_l}{\partial x_j} - \frac{\partial A_j}{\partial x_l}\right)\widehat{\alpha}_j\widehat{\alpha}_l,\\ D(\widehat{\mathcal{D}}^2(A; k+\mathrm{i}\varkappa e)) &= D(\widehat{H}(A; k+\mathrm{i}\varkappa e)\otimes\widehat{I}_M) = \widetilde{H}^2(K; \mathbb{C}^M).\end{aligned} \tag{52}$$

For all vector functions $\phi \in \widetilde{H}^1(K; \mathbb{C}^M)$,

$$\widehat{\mathcal{D}}(0; k+\mathrm{i}\varkappa e)\phi = \sum_{N\in\Lambda^*}\widehat{\mathcal{D}}_N(k;\varkappa)\phi_N\,\mathrm{e}^{2\pi\mathrm{i}(N,x)},$$

where

$$\widehat{\mathcal{D}}_N(k;\varkappa) = \sum_{j=1}^{n}(k_j + 2\pi N_j + \mathrm{i}\varkappa e_j)\widehat{\alpha}_j, \qquad N_j = (N, \mathcal{E}_j), \qquad j = 1, \ldots, n.$$

In the following, we shall use the notation $\widehat{\mathbb{G}}^\zeta_\pm = \widehat{\mathbb{G}}^\zeta_\pm(k+\mathrm{i}\varkappa e) = \widehat{G}^\zeta_\pm \otimes \widehat{I}_M$, $\zeta \in \mathbb{C}$ (and $\widehat{\mathbb{G}}_\pm \doteq \widehat{\mathbb{G}}^1_\pm$):

$$D(\widehat{\mathbb{G}}^\zeta_\pm) = \begin{cases}\widetilde{H}^{\mathrm{Re}\,\zeta}(K;\mathbb{C}^M) & \text{if} \quad \mathrm{Re}\,\zeta > 0,\\ L^2(K;\mathbb{C}^M) & \text{if} \quad \mathrm{Re}\,\zeta \leqslant 0.\end{cases}$$

Let $\widehat{\mathbb{L}} = \widehat{\mathbb{L}}(k+\mathrm{i}\varkappa e) = \widehat{\mathbb{G}}_+\widehat{\mathbb{G}}_-$, then $\widehat{\mathbb{L}}^{1/2} = \widehat{\mathbb{L}}^{1/2}(k+\mathrm{i}\varkappa e) = \widehat{\mathbb{G}}_+^{1/2}\widehat{\mathbb{G}}_-^{1/2}$.

For all $k \in \mathbb{R}^n$, all $\varkappa \geqslant 0$, and all $N \in \Lambda^*$, the inequalities

$$G^-_N(k;\varkappa)\|u\| \leqslant \|\widehat{\mathcal{D}}_N(k;\varkappa)u\| \leqslant G^+_N(k;\varkappa)\|u\|, \qquad u \in \mathbb{C}^M,$$

hold. Hence, for all vector functions $\phi \in \widetilde{H}^1(K;\mathbb{C}^M)$,

$$\|\widehat{\mathbb{G}}_-\phi\| \leqslant \|\widehat{\mathcal{D}}(0; k+\mathrm{i}\varkappa e)\phi\| \leqslant \|\widehat{\mathbb{G}}_+\phi\|.$$

For vectors $\widetilde{e} \in S_{n-2}(e)$, define the orthogonal projections on $\mathbb{C}^M$

$$\widehat{P}^\pm_{\widetilde{e}} = \frac{1}{2}\left(\widehat{I} \mp \mathrm{i}\left(\sum_{j=1}^{n}e_j\widehat{\alpha}_j\right)\left(\sum_{j=1}^{n}\widetilde{e}_j\widehat{\alpha}_j\right)\right).$$

We write $\widetilde{e}(y) \doteq |y_\perp|^{-1}y_\perp \in S_{n-2}(e)$ for vectors $y \in \mathbb{R}^n$ with $y_\perp \neq 0$.

If $k \in \mathbb{R}^n$, $N \in \Lambda^*$, and $k_\perp + 2\pi N_\perp \neq 0$, then

$$\widehat{P}^{\pm}_{\widetilde{e}(k+2\pi N)} \widehat{\mathcal{D}}_N(k;\varkappa) \widehat{P}^{\pm}_{\widetilde{e}(k+2\pi N)} = \widehat{O}_M \tag{53}$$

(where $\widehat{O}_M$ is the zero $M \times M$-matrix) and, for all vectors $u \in \mathbb{C}^M$ (and all $\varkappa \geqslant 0$),

$$\left\| \widehat{\mathcal{D}}_N(k;\varkappa) \widehat{P}^{\pm}_{\widetilde{e}(k+2\pi N)} u \right\| = G^{\pm}_N(k + \mathrm{i}\varkappa e) \left\| \widehat{P}^{\pm}_{\widetilde{e}(k+2\pi N)} u \right\|. \tag{54}$$

If $k_\perp + 2\pi N_\perp = 0$, then $G^+_N(k + \mathrm{i}\varkappa e) = G^-_N(k + \mathrm{i}\varkappa e)$.

Let $\mathfrak{K}(\gamma)$ be the set of vectors $k \in \mathbb{R}^n$ such that $k_\perp + 2\pi N_\perp \neq 0$ for all $N \in \Lambda^*$; $\mathfrak{K}_\pi(\gamma) \doteq \mathfrak{K}(\gamma) \cap \{k \in \mathbb{R}^n : |(k,\gamma)| = \pi\}$.

Given $k \in \mathfrak{K}(\gamma)$, denote by $\widehat{P}^{\pm} = \widehat{P}^{\pm}(k;e)$ the orthogonal projections on $L^2(K;\mathbb{C}^M)$

$$\widehat{P}^{\pm}\phi = \sum_{N \in \Lambda^*} \widehat{P}^{\pm}_{\widetilde{e}(k+2\pi N)} \phi_N \, \mathrm{e}^{2\pi \mathrm{i}(N,x)}, \qquad \phi \in L^2(K;\mathbb{C}^M).$$

Since $\widehat{P}^+ + \widehat{P}^- = \widehat{I}$ (where $\widehat{I}$ is the identity operator on $L^2(K;\mathbb{C}^M)$), from (53) and (54) it follows that

$$\|\widehat{P}^{\pm}\widehat{\mathcal{D}}(0;k+\mathrm{i}\varkappa e)\phi\| = \|\widehat{\mathbb{G}}_{\mp}\widehat{P}^{\mp}\phi\|,$$
$$\|\widehat{\mathcal{D}}(0;k+\mathrm{i}\varkappa e)\phi\|^2 = \|\widehat{\mathbb{G}}_-\widehat{P}^-\phi\|^2 + \|\widehat{\mathbb{G}}_+\widehat{P}^+\phi\|^2, \qquad \phi \in \widetilde{H}^1(K;\mathbb{C}^M).$$

Theorem 3.1 (see [46])**.** *Let $n \geqslant 3, \mathfrak{a} \geqslant 0, \Theta \in [0,1)$, and $R \geqslant 0$. Suppose $A \in L^2(K;\mathbb{R}^n)$, $A_0 = 0$ and (for the magnetic potential A) the conditions (A_1) and (A_2) are satisfied for a vector $\gamma \in \Lambda\backslash\{0\}$ and a measure $\mu \in \mathfrak{M}_h, h > 0$, and, moreover, $\||A|\|_{2,\gamma} \leqslant \mathfrak{a}, \theta(\Lambda,\gamma,h,\mu;A) \leqslant \Theta$, and $A_N = 0$ for all vectors $N \in \Lambda^*$ with $2\pi|N_\perp| > R$. Then there exists a constant $\widetilde{C}_1 = \widetilde{C}_1(n,\Lambda,|\gamma|,h,\|\mu\|;\mathfrak{a},\Theta) \in (0,1)$ such that for every $\delta \in (0,1)$ there is a number $\widetilde{a} = \widetilde{a}(\widetilde{C}_1;\delta,R) \in (0,\widetilde{C}_1]$ such that for any $a \in (0,\widetilde{a}]$, the estimate*

$$\|(\widehat{P}^+ + a\widehat{P}^-)\widehat{\mathcal{D}}(A;k+\mathrm{i}\varkappa e)\phi\|^2 \geqslant (1-\delta)\|(\widetilde{C}_1\widehat{\mathbb{G}}_-\widehat{P}^- + a\widehat{\mathbb{G}}_+\widehat{P}^+)\phi\|^2 \tag{55}$$

holds for all vectors $k \in \mathfrak{K}_\pi(\gamma)$, all vector functions $\phi \in \widetilde{H}^1(K;\mathbb{C}^M)$, and all sufficiently large numbers $\varkappa \geqslant \varkappa_0 > 0$ (where $\varkappa_0$ depends on the number a but does not depend on k and ϕ).

Remark 5. In [46], theorem 3.1 was formulated for the case $a = \widetilde{a}$. But in the proof of theorem 3.1, only upper bounds for the number $\widetilde{a}$ were used. Hence, theorem 3.1 is also true for all $a \in (0,\widetilde{a}]$ (nevertheless the number $\varkappa_0$ depends on the number a).

Under the conditions of theorem 3.1, instead of the vector $\gamma \in \Lambda\backslash\{0\}$ one can pick the vector $-\gamma$ (without change of the basis vectors $\mathcal{E}_j, j = 1,\ldots,n$). Then the following changes are to be made: $e \to -e, k_\parallel \to -k_\parallel, k_\perp \to k_\perp, N_\parallel \to -N_\parallel, N_\perp \to N_\perp$ (for all $k \in \mathbb{R}^n$ and all $N \in \Lambda^*$). Furthermore, the numbers $G^{\pm}_N(k;\varkappa)$, the sets $\mathfrak{K}_\pi(\gamma)$, and the vectors $\widetilde{e}(k+2\pi N)$ do not change, but the orthogonal projections $\widehat{P}^+$ and $\widehat{P}^-$ are replaced by the orthogonal projections $\widehat{P}^-$ and $\widehat{P}^+$, respectively. Therefore, for any $a \in (0,\widetilde{a}]$ and for all vectors $k \in \mathfrak{K}_\pi(\gamma)$, all vector functions $\phi \in \widetilde{H}^1(K;\mathbb{C}^M)$, and all sufficiently large numbers $\varkappa \geqslant \varkappa_0 > 0$ (where $\varkappa_0$ does not depend on k and ϕ), the estimate

$$\|(\widehat{P}^- + a\widehat{P}^+)\widehat{\mathcal{D}}(A;k-\mathrm{i}\varkappa e)\phi\|^2 \geqslant (1-\delta)\|(\widetilde{C}_1\widehat{\mathbb{G}}_-\widehat{P}^+ + a\widehat{\mathbb{G}}_+\widehat{P}^-)\phi\|^2 \tag{56}$$

is also valid.

For vector functions $\phi \in L^2(K;\mathbb{C}^M)$, we deduce from (55) and (56) that

$$\left\|(\widehat{P}^+ + a\widehat{P}^-)\widehat{\mathcal{D}}(A;k+\mathrm{i}\varkappa e)\left(\widetilde{C}_1^{-1}\widehat{\mathbb{G}}_-^{-1}\widehat{P}^- + a^{-1}\widehat{\mathbb{G}}_+^{-1}\widehat{P}^+\right)\phi\right\|^2 \geqslant (1-\delta)\|\phi\|^2 \tag{57}$$

and

$$\left\|(\widehat{P}^- + a\widehat{P}^+)\widehat{\mathcal{D}}(A; k - \mathrm{i}\varkappa e)\left(\widetilde{C}_1^{-1}\widehat{\mathbb{G}}_-^{-1}\widehat{P}^+ + a^{-1}\widehat{\mathbb{G}}_+^{-1}\widehat{P}^-\right)\phi\right\|^2 \geqslant (1-\delta)\|\phi\|^2, \tag{58}$$

respectively. Since the norm of a bounded linear operator acting on the Hilbert space is equal to the norm of the adjoint operator, we get from the last estimate that for all $\phi \in \widetilde{H}^1(K; \mathbb{C}^M)$

$$\left\|\left(\widetilde{C}_1^{-1}\widehat{\mathbb{G}}_-^{-1}\widehat{P}^+ + a^{-1}\widehat{\mathbb{G}}_+^{-1}\widehat{P}^-\right)\widehat{\mathcal{D}}(A; k + \mathrm{i}\varkappa e)(\widehat{P}^- + a\widehat{P}^+)\phi\right\|^2 \geqslant (1-\delta)\|\phi\|^2. \tag{59}$$

The following inequality is a direct consequence of (57) and (59):

$$\|(\widehat{\mathbb{G}}_-^{-1}\widehat{P}^+ + \widetilde{C}_1 a^{-1}\widehat{\mathbb{G}}_+^{-1}\widehat{P}^-)\widehat{\mathcal{D}}^2(A; k + \mathrm{i}\varkappa e)(\widehat{\mathbb{G}}_+^{-1}\widehat{P}^+ + \widetilde{C}_1^{-1}a\widehat{\mathbb{G}}_-^{-1}\widehat{P}^-)\phi\|$$
$$\geqslant \widetilde{C}_1(1-\delta)\|\phi\|, \qquad \phi \in \widetilde{H}^1(K; \mathbb{C}^M). \tag{60}$$

The inequality (60) plays a key role in the proof of theorem 1.3.

In the following, we assume that $\delta = \frac{1}{6}$. By (57) and (58), it follows that $\operatorname{Ker}\widehat{\mathcal{D}}(A; k + \mathrm{i}\varkappa e) = \operatorname{Coker}\widehat{\mathcal{D}}(A; k + \mathrm{i}\varkappa e) = \{0\}$. Hence for the range of the operator $\widehat{\mathcal{D}}(A; k + \mathrm{i}\varkappa e)$, we have $R(\widehat{\mathcal{D}}(A; k + \mathrm{i}\varkappa e)) = L^2(K; \mathbb{C}^M)$.

Let us denote

$$\widehat{\mathcal{B}}(A) = \frac{\mathrm{i}}{2}\sum_{j\neq l}\left(\frac{\partial A_l}{\partial x_j} - \frac{\partial A_j}{\partial x_l}\right)\widehat{\alpha}_j\widehat{\alpha}_l.$$

The estimate

$$\left\|\widehat{\mathbb{G}}_-^{-1}\widehat{P}^+\widehat{\mathcal{B}}(A)\widetilde{C}_1^{-1}a\widehat{\mathbb{G}}_-^{-1}\widehat{P}^-\phi\right\|$$
$$\leqslant \frac{n(n-1)}{2}\frac{|\gamma|^2}{\pi^2}\widetilde{C}_1^{-1}a\left(\max_{x\in K, l\neq j}\left|\frac{\partial A_l}{\partial x_j}\right|\right)\|\phi\|, \qquad \phi \in L^2(K; \mathbb{C}^M),$$

holds. We choose (and fix) a number $a \in (0, \widetilde{a}]$ such that

$$\frac{n(n-1)}{2}\frac{|\gamma|^2}{\pi^2}\widetilde{C}_1^{-1}a\left(\max_{x\in K, l\neq j}\left|\frac{\partial A_l}{\partial x_j}\right|\right) \leqslant \frac{1}{6}\widetilde{C}_1.$$

Then there is a sufficiently large number $\varkappa_0 > 0$ such that for all $\varkappa \geqslant \varkappa_0$, all $k \in \mathfrak{K}_\pi(\gamma)$, and all $\phi \in L^2(K; \mathbb{C}^M)$

$$\left\|\left(\widehat{\mathbb{G}}_-^{-1}\widehat{P}^+ + \widetilde{C}_1 a^{-1}\widehat{\mathbb{G}}_+^{-1}\widehat{P}^-\right)\widehat{\mathcal{B}}(A)\left(\widehat{\mathbb{G}}_+^{-1}\widehat{P}^+ + \widetilde{C}_1^{-1}a\widehat{\mathbb{G}}_-^{-1}\widehat{P}^-\right)\phi\right\| \leqslant \tfrac{1}{3}\widetilde{C}_1\|\phi\|.$$

Consequently, by (52) and (60), it follows that

$$\left\|\left(\widehat{\mathbb{G}}_-^{-1}\widehat{P}^+ + \widetilde{C}_1 a^{-1}\widehat{\mathbb{G}}_+^{-1}\widehat{P}^-\right)(\widehat{H}(A; k + \mathrm{i}\varkappa e)\otimes\widehat{I}_M)\left(\widehat{\mathbb{G}}_+^{-1}\widehat{P}^+ + \widetilde{C}_1^{-1}a\widehat{\mathbb{G}}_-^{-1}\widehat{P}^-\right)\phi\right\|$$
$$\geqslant \tfrac{1}{2}\widetilde{C}_1\|\phi\|, \qquad \phi \in \widetilde{H}^1(K; \mathbb{C}^M). \tag{61}$$

Since the choice of the matrices $\widehat{\alpha}_j$, $j = 1, \ldots, n$, is not specified, we can replace the matrix $\widehat{\alpha}_1$ by the matrix $-\widehat{\alpha}_1$ (the commutation relations (51) do not change under such replacement). Then the orthogonal projections $\widehat{P}^+$ and $\widehat{P}^-$ substitute each other, and we obtain from (61) that

$$\left\|\left(\widehat{\mathbb{G}}_-^{-1}\widehat{P}^- + \widetilde{C}_1 a^{-1}\widehat{\mathbb{G}}_+^{-1}\widehat{P}^+\right)(\widehat{H}(A; k + \mathrm{i}\varkappa e)\otimes\widehat{I}_M)\left(\widehat{\mathbb{G}}_+^{-1}\widehat{P}^- + \widetilde{C}_1^{-1}a\widehat{\mathbb{G}}_-^{-1}\widehat{P}^+\right)\phi\right\|$$
$$\geqslant \tfrac{1}{2}\widetilde{C}_1\|\phi\|, \qquad \phi \in \widetilde{H}^1(K; \mathbb{C}^M). \tag{62}$$

Inequalities (61) and (62) imply that $\operatorname{Ker}\widehat{H}(A; k+\mathrm{i}\varkappa e)\otimes\widehat{I}_M = \operatorname{Coker}\widehat{H}(A; k+\mathrm{i}\varkappa e)\otimes\widehat{I}_M = \{0\}$ and $R(\widehat{H}(A; k + \mathrm{i}\varkappa e)\otimes\widehat{I}_M) = L^2(K; \mathbb{C}^M)$. Hence,

$$\operatorname{Ker}\widehat{H}(A; k + \mathrm{i}\varkappa e) = \operatorname{Coker}\widehat{H}(A; k + \mathrm{i}\varkappa e) = \{0\}$$

(and $D(\widehat{H}(A; k + \mathrm{i}\varkappa e) = \widetilde{H}^2(K)$, $R(\widehat{H}(A; k + \mathrm{i}\varkappa e) = L^2(K))$.

Now let us rewrite inequalities (61) and (62) in the form

$$\left\|(\widehat{\mathbb{G}}_+\widehat{P}^+ + \widetilde{C}_1 a^{-1}\widehat{\mathbb{G}}_-\widehat{P}^-)(\widehat{H}^{-1}(A;k+\mathrm{i}\varkappa e)\otimes\widehat{I}_M)\big(\widehat{\mathbb{G}}_-\widehat{P}^+ + \widetilde{C}_1^{-1}a\widehat{\mathbb{G}}_+\widehat{P}^-\big)\phi\right\| \leqslant 2\widetilde{C}_1^{-1}\|\phi\|, \qquad \phi\in\widetilde{H}^1(K;\mathbb{C}^M), \tag{63}$$

$$\left\|(\widehat{\mathbb{G}}_+\widehat{P}^- + \widetilde{C}_1 a^{-1}\widehat{\mathbb{G}}_-\widehat{P}^+)(\widehat{H}^{-1}(A;k+\mathrm{i}\varkappa e)\otimes\widehat{I}_M)\big(\widehat{\mathbb{G}}_-\widehat{P}^- + \widetilde{C}_1^{-1}a\widehat{\mathbb{G}}_+\widehat{P}^+\big)\phi\right\| \leqslant 2\widetilde{C}_1^{-1}\|\phi\|, \qquad \phi\in\widetilde{H}^1(K;\mathbb{C}^M). \tag{64}$$

For all $\zeta\in\mathbb{C}$ (and for fixed $\varkappa\geqslant\varkappa_0$, $k\in\mathfrak{K}_\pi(\gamma)$ and $a\in(0,\widetilde{a}]$) define the operators

$$\widehat{\mathcal{Q}}(\zeta) = \big(\widehat{\mathbb{G}}_+^{1-\zeta}(\widetilde{C}_1a^{-1})^{\zeta}\widehat{\mathbb{G}}_-^{\zeta}\widehat{P}^+ + \widehat{\mathbb{G}}_+^{\zeta}(\widetilde{C}_1a^{-1})^{1-\zeta}\widehat{\mathbb{G}}_-^{1-\zeta}\widehat{P}^-\big) \times(\widehat{H}^{-1}(A;k+\mathrm{i}\varkappa e)\otimes\widehat{I}_M)\big(\widehat{\mathbb{G}}_-^{1-\zeta}\big(\widetilde{C}_1^{-1}a\big)^{\zeta}\widehat{\mathbb{G}}_+^{\zeta}\widehat{P}^+ + \widehat{\mathbb{G}}_-^{\zeta}\big(\widetilde{C}_1^{-1}a\big)^{1-\zeta}\widehat{\mathbb{G}}_+^{1-\zeta}\widehat{P}^-\big),$$

$D(\widehat{\mathcal{Q}}(\zeta)) = \widetilde{H}^1(K;\mathbb{C}^M)\subset L^2(K;\mathbb{C}^M)$. For all $\phi\in\widetilde{H}^1(K;\mathbb{C}^M)$, the function $\mathbb{C}\ni\zeta\to\widehat{\mathcal{Q}}(\zeta)\phi\in L^2(K;\mathbb{C}^M)$ is uniformly bounded for $0\leqslant\operatorname{Re}\zeta\leqslant 1$ (see (63) and (64)) and analytic for $0<\operatorname{Re}\zeta<1$. If $\operatorname{Re}\zeta=0$ or $\operatorname{Re}\zeta=1$, then (63) and (64) imply that

$$\|\widehat{\mathcal{Q}}(\zeta)\phi\|\leqslant 2\widetilde{C}_1^{-1}\|\phi\|. \tag{65}$$

Therefore estimate (65) is true for all $\zeta\in\mathbb{C}$ with $0\leqslant\operatorname{Re}\zeta\leqslant 1$. In particular, for $\zeta=\frac{1}{2}$, we have

$$\|\widehat{\mathbb{L}}^{1/2}(\widehat{H}^{-1}(A;k+\mathrm{i}\varkappa e)\otimes\widehat{I}_M)\widehat{\mathbb{L}}^{1/2}\phi\|\leqslant 2\widetilde{C}_1^{-1}\|\phi\|, \qquad \phi\in\widetilde{H}^1(K;\mathbb{C}^M),$$

and hence for all $\varkappa\geqslant\varkappa_0$, all $k\in\mathfrak{K}_\pi(\gamma)$, and all $\phi\in\widetilde{H}^1(K)$

$$\|\widehat{L}^{1/2}\widehat{H}^{-1}(A;k+\mathrm{i}\varkappa e)\widehat{L}^{1/2}\phi\|\leqslant 2\widetilde{C}_1^{-1}\|\phi\|.$$

Whence

$$\|\widehat{L}^{-1/2}\widehat{H}(A;k+\mathrm{i}\varkappa e)\widehat{L}^{-1/2}\phi\|\geqslant \tfrac{1}{2}\widetilde{C}_1\|\phi\|, \qquad \phi\in\widetilde{H}^1(K). \tag{66}$$

By continuity, the last estimate extends to all vectors $k\in\mathbb{R}^n$ with $|(k,\gamma)|=\pi$. Finally, let $C_1=\frac{1}{2}\widetilde{C}_1$. Then estimate (10) follows from (66) for all $\varkappa\geqslant\varkappa_0$, all vectors $k\in\mathbb{R}^n$ with $|(k,\gamma)|=\pi$, and all functions $\phi\in\widetilde{H}^2(K)$. Since the set $\widetilde{H}^2(K)$ is dense in $\widetilde{H}^1(K)$ and the form $W(A;k+\mathrm{i}\varkappa e;\psi,\phi)$ is continuous in functions ψ and ϕ from the Sobolev class $\widetilde{H}^1(K)$, estimate (10) is also valid for all functions $\phi\in\widetilde{H}^1(K)$. This completes the proof of theorem 1.3.

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